Department of Applied Mathematics and Statistics
The Johns Hopkins University

# INTRODUCTORY EXAMINATION–SPRING SESSION

January 22, 2009

**Instructions: Read carefully!**

1. This **closed-book** examination consists of 15 problems, each worth 5 points. The passing grade has been set at 50 points, i.e. 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.
3. The problems have not been grouped by topic, but there are roughly equally many mainly motivated by each of the three areas identified in the syllabus (linear algebra; real analysis; probability;). Nor have the problems been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.
5. The examination will begin at 8:30 AM; lunch and refreshments will be provided. The exam will end just before 5:00 PM. You may leave before then, but in that case you may not return.
6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
7. **No calculators of any sort are needed or permitted.**

1. Let $A$ be a Hermitian, tridiagonal, square matrix of order $n$, with all its subdiagonals and superdiagonals nonzero (i.e., $A = (a_{ij}, i, j = 1, \ldots, n)$ is such that $a_{ij} = 0$ if $|i - j| > 1$ and $a_{ij} \neq 0$ if $|i - j| = 1$). Prove that the eigenvalues of $A$ are distinct.

   *Solution:* By spectral theorem, all eigenvalues of $A$ are real, and for every eigenvalue $\lambda$, the algebraic multiplicity of $\lambda$ is the same as the geometric multiplicity of $\lambda$. Thus we need to show that the geometric multiplicity of $\lambda$ is 1, and it sufficies to prove dim(kernel($A - \lambda I$))$\leq$ 1. Suppose not. Then $\exists$ a linear subspace $V$ with dim($V$)= 2 s.t. $(A - \lambda I)v = 0$, $\forall v \in V$. Let $W := \{[0, x_2, x_3, \ldots, x_n]^T : x_j \in \mathbf{C}, j = 2, \ldots, n\}$. Then since both $V$ and $W$ are subspaces of $\mathbf{C}^n$, dim($V + W$)$\leq n$. On the other hand, the dimension of sum of two subspaces is given by

   $$dim(V + W) = dim(V) + dim(W) - dim(V \cap W)$$

   and dim($W$)= $n - 1$. Thus dim($V \cap W$)$\geq$ 1, i.e. $\exists v \neq 0$, s.t. $v \in V \cap W$. Let the first nonzero component of $v$ be $v_J$. Then $J \geq 2$ since $v \in W$. Furthermore, since $v \in V$, $(A - \lambda I)v = 0$. On the other hand, the direct computation of $(A - \lambda I)v$ shows that its $(J - 1)$-th component is $c \cdot v_J$ for some nonzero constant $c$. Therefore we obtained $0 = c \cdot v_J \neq 0$ which is a contradiction.

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2. Let $A = \left(a_{ij}\right)_{i,j=1,\ldots,n}$ be the $n \times n$ matrix whose entries satisfy

   $$a_{ij} = \begin{cases} 1 & \text{if } j \geq i \\ -1 & \text{if } j = i - 1 \\ 0 & \text{otherwise,} \end{cases}$$

   so that, for example, if $n = 3$ the matrix takes the form

   $$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

   What is $\det(A)$? Justify your answer rigorously.

   *Solution:* Let $A_n$ denote the matrix in the $n \times n$ case. Observe that when eliminate the first column and first row, or the first column and the second row of $A_n$, we obtain $A_{n-1}$. Thus, expanding on the first column we obtain

   $$\det A_n = 1 \times \det A_{n-1} - (-1) \det A_{n-1} = 2 \det A_{n-1}$$

   and since $\det A_2$ is clearly 2, we see by induction that $\det A_n = 2^{n-1}$.

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3. Let $f$ be a $k$ times differentiable real-valued function on a nonempty finite open interval $(a,b)$, $k$ being a positive integer. Show that if the $k$-th derivative $f^{(k)}$ is bounded on $(a,b)$, then $f$ is uniformly continuous on $(a,b)$.

*Solution:* We first note that if $f'$ is bounded, say $|f'(x)| \le M$ for all $x \in (a,b)$, then $f$ is uniformly continuous, because it follows from the mean-value theorem that for any $x$, $y \in (a,b)$, we have

$$|f(x) - f(y)| = |f'(z)(x-y)| \le M|x-y|.$$

Then we show that if $f''$ is bounded then so is $f'$. Let $|f''(x)| \le M$ for all $x \in (a,b)$ and let $x_0$ be any fixed point in $(a,b)$. Then we have

$$|f'(x)| \le |f'(x_0)| + |f'(x) - f'(x_0)|.$$

It follows from the mean-valued theorem again that

$$|f'(x)| \le |f'(x_0)| + M|x-x_0| \le |f'(x_0)| + M(b-a).$$

We repeat the argument to deduce the boundedness of $f^{(k-1)}, \cdots, f'$ from the boundedness of $f^{(k)}$.

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4. Let $f : [0,1] \to [0,1]$ be continuous. Suppose there is a value $a \in [0,1]$ such that $f[f(a)] = a$. Prove there is a value $b \in [0,1]$ (not necessarily different from $a$) such that $f(b) = b$.

*Solution:* Suppose, for contradiction, no such value $b$ exists. From the intermediate value theorem we can conclude that either for all $x \in [0,1]$ we have $f(x) - x > 0$ or else for all $x$, $f(x) - x < 0$, i.e., $f(x) > x$ always or $f(x) < x$ always (for $x \in [0,1]$).

In the first case we have

$$f(f(a)) > f(a) > a$$

and in the second case

$$f(f(a)) < f(a) < a$$

both contradicting $f(f(a)) = a$. Therefore there is such a $b$.

Note: The problem can be rephrased with $f : \mathbf{R} \to \mathbf{R}$ if that's preferable.

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5. Suppose that $\mathbf{h} = (\alpha, \beta, \gamma)$ is a smooth, 1-1 function from $V \subset \mathbb{R}^3$ onto $W \subset \mathbb{R}^3$, where $V, W$ are open neighborhoods. Use the coordinate functions to define two vector fields on $V$ :

$$\mathbf{f}(\mathbf{x}) = \nabla\alpha(\mathbf{x}) \times \nabla\beta(\mathbf{x}),\ \mathbf{g}(\mathbf{x}) = \nabla\gamma(\mathbf{x})$$

Prove that

$$\int_V \mathbf{f}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x})\, d\mathbf{x} = \pm \int_W d\mathbf{y}.$$

*Solution:* Recall that, for three vectors, $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$, we have the identity

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \det(\mathbf{a}, \mathbf{b}, \mathbf{c}).$$

Using this, we get

$$\mathbf{f}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x}) = \det \frac{\partial \mathbf{h}}{\partial \mathbf{x}},$$

which is the Jacobian of transformation. The equality of the two integrals follows by the change of variables formula, with the sign being the sign of the determinant.

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6. Let $X, Y$ be independent random variables, each with density $f(x) = x^{-2}, x \geq 1$ and $f(x) = 0$ otherwise. Calculate the probability density function of the vector $(U, V)$ where

$$U = XY,\ V = \frac{X}{Y}.$$

*Solution:* Define $\phi(x,y) = (xy, \frac{x}{y})$ for $x, y \geq 1$. The inverse mapping is $\phi^{-1}(u,v) = ((uv)^{1/2}, (\frac{u}{v})^{1/2})$ defined for $uv \geq 1, u \geq v, u \geq 1, v \geq 0$. The magnitude of the Jacobian of $\phi^{-1}$ is $\frac{1}{2v}$. Hence,

$$\begin{aligned} f_{U,V}(u,v) &= f_{X,Y}(\phi^{-1}(u,v))\frac{1}{2v} \\ &= I_{\{(uv)^{1/2} \geq 1\}} \frac{1}{uv} I_{\{(\frac{u}{v})^{1/2} \geq 1\}} \left(\frac{v}{u}\right)\left(\frac{1}{2v}\right),\ u \geq 1, v \geq 0 \\ &= \frac{1}{2u^2 v} I_{\{uv \geq 1\}} I_{\{u \geq v\}},\ u \geq 1, v \geq 0. \end{aligned}$$

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7. Let $A$ and $B$ be two matrices such that $B-A$, $A^{-1}$ and $B^{-1}$ all have positive coefficients. Show that $A^{-1}-B^{-1}$ also has positive coefficients.

   *Solution:* The result is obvious from the decomposition

   $$A^{-1}-B^{-1}=B^{-1}(B-A)A^{-1}.$$

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8. Let $X$ and $Y$ be two random variables with respective probability density functions $f_X$ and $f_Y$, $f_X$ and $f_Y$ being continuous functions defined on $\mathbb{R}$.

   Prove that, if $f_X(x)>0 \Leftrightarrow f_Y(x)>0$ for $x \in \mathbb{R}$, then, $P(X \in I)>0 \Leftrightarrow P(Y \in I)>0$ for any interval $I$ in $\mathbb{R}$.

   Is the converse statement also true? Justify your answer.

   *Solution:* Assume that $P(X \in I)>0$, and that $P(Y \in I)=0$. Since $f_Y$ is continuous, this is only possible if $f_Y(x)=0$ for all $x \in I$, which implies, by assumption that $f_X(x)=0$ on $I$, so that $P(X \in I)=0$, which is a contradiction.

   The converse statement is false. One of the densities can vanish at an isolated point while the other is positive, while still ensuring that $P(X \in I)>0 \leftrightarrow P(Y \in I)>0$. A counter-example can be easily made up with piecewise linear densities.

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9. Your two friends are to meet you at a particular time. The friends independently arrive late, by random amounts of time, $X$ and $Y$, which are exponentially distributed with a mean of five minutes. What is the expected waiting time until the second friend arrives?

   *Solution:* The arrival of the first friend occurs at $Z=\min\{X,Y\}$. Since $X$ and $Y$ are independent exponentials, $Z$ is exponential, with its parameter equal to the sum of the parameters of $X$ and $Y$. Since the mean of $X$ and $Y$ is 5, their parameter values are $\frac{1}{5}$. Thus, the parameter of $Z$ is $\frac{2}{5}$, and its mean is $\frac{5}{2}$.

   By the lack of memory property of the exponential distribution, the additional time after $Z$ until the second friend arrives is exponentially distributed with mean 5.

   The expected waiting time until the second friend arrives is the sum, which is 7.5 minutes.

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10. Let $n$ be a fixed positive integer, and suppose that $X$ and $\Theta$ are random variables such that the conditional distribution of $X$ given $\Theta = p$ is a binomial with parameters $n$ and $p$. Further assume that $\Theta$ has a beta distribution with parameters $\alpha$ and $\beta$ ($\alpha, \beta > 0$), i.e., $\Theta$ has the following p.d.f.:

$$f_\Theta(p) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1} & \text{if } 0 < p < 1 \\ 0 & \text{elsewhere} \end{cases} .$$

Determine the conditional distribution of $\Theta$ given $X = x$.

*Solution:* We are told $f_{X|\Theta}(x|p) = \binom{n}{x} p^x (1-p)^{n-x}$ for $x = 0, 1, \ldots, n$. Therefore, by the Baye's rule

$$\begin{aligned} f_{\Theta|X}(p|x) &= \frac{f_{X|\Theta}(x|p)\, f_\Theta(p)}{\int_0^1 f_{X|\Theta}(x|y)\, f_\Theta(y)\, dy} \\ &= \frac{p^x(1-p)^{n-x} \cdot p^{\alpha-1}(1-p)^{\beta-1}}{\int_0^1 y^x(1-y)^{n-x} \cdot y^{\alpha-1}(1-y)^{\beta-1}\, dy} \\ &= \frac{p^{x+\alpha-1}(1-p)^{n-x+\beta-1}}{\int_0^1 y^{x+\alpha-1}(1-y)^{n-x+\beta-1}\, dy} \\ &= \frac{\Gamma(n+\alpha+\beta)}{\Gamma(x+\alpha)\Gamma(n-x+\beta)} p^{x+\alpha-1}(1-p)^{n-x+\beta-1} \end{aligned}$$

and it follows $\Theta|X = x$ has a beta distribution with parameters $x+\alpha, n-x+\beta$.

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11. Let $a, b, c, d$ be distinct real numbers. Consider the four points in $\mathbb{R}^3$:

$$A = \begin{bmatrix} a \\ a^2 \\ a^3 \end{bmatrix}, \quad B = \begin{bmatrix} b \\ b^2 \\ b^3 \end{bmatrix}, \quad C = \begin{bmatrix} c \\ c^2 \\ c^3 \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} d \\ d^2 \\ d^3 \end{bmatrix}.$$

Prove that lines $AB$ and $CD$ do not intersect.

*Solution:* The volume of the tetrahedron spanned by the four points is

$$\frac{1}{6} \det \begin{bmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{bmatrix}.$$

If the lines $AB$ and $CD$ intersected, this would be zero. But the matrix is a Vandermonde matrix and so its determinant is

$$(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$$

which is not zero because the four numbers $a, b, c, d$ are distinct.

Therefore the lines do not intersect.

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12. Let $S$ and $T$ be two subspaces in $R^n$ and $A$ be a $n \times n$ real matrix. Show that if $\dim(S) > \dim(T)$ and $Ax \in T$ for any $x \in S$ then there exists a nonzero $x \in S$ such that $Ax = 0$.

*Solution:* Let $k = \dim(S)$ and $\{x_1, x_2, \cdots, x_k\}$ be a set of linearly independent vectors in $S$. It follows that $\dim(T) < k$ and $Ax_i \in T$ that $\{Ax_1, \cdots, Ax_k\}$ must be linearly dependent. Therefore, there exist $\alpha_1, \cdots, \alpha_k$, not all zeros, such that

$$\alpha_1 Ax_1 + \cdots + \alpha_k Ax_k = 0.$$

Then the vector $y = \alpha_1 x_1 + \cdots + \alpha_k x_k$ is a nonzero vector in $S$ and $Ay = 0$.

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13. In a laboratory, two types of insects are generated every day. Each new insect is equally likely to be one of two possible types. Assume that, on any given day, the total number of insects generated is Poisson distributed with parameter $\lambda$. For a certain day, let $N_i$ denote the number of insects of type $i$ that are generated ($i = 1, 2$). What is the probability that $N_1 > 0$ and $N_2 = 0$? Express your answer as simply as possible in terms of $\lambda$.

[HINT: One solution strategy is to condition on the value of $N := N_1 + N_2$.]

*Solution:* A student may know by "Poisson thinning" that $N_1$ and $N_2$ are independent and each distributed Poisson($\lambda/2$). Then the proof is very easy:

$$P\{N_1 > 0, N_2 = 0\} = P\{N_1 > 0\}P\{N_2 = 0\} = (1 - e^{-\lambda/2})e^{-\lambda/2} = e^{-\lambda/2} - e^{-\lambda}.$$

*Solution # 2:* We follow the hint and condition on $N$ [which is distributed Poisson($\lambda$)]:

$$\begin{aligned} P\{N_1 > 0, N_2 = 0\} &= P\{N > 0, N_2 = 0\} = \sum_{n=1}^{\infty} P\{N = n, N_2 = 0\} \\ &= \sum_{n=1}^{\infty} P\{N = n\} P\{N_2 = 0 \,|\, N = n\} \\ &= \sum_{n=1}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} 2^{-n} \\ &= e^{-\lambda}\left(e^{\lambda/2} - 1\right) = e^{-\lambda/2} - e^{-\lambda}. \end{aligned}$$

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14. Let $(c_n)$ be a decreasing sequence of positive numbers. If the series $\sum(c_n \sin nx)$ is uniformly convergent for $x \in \mathbb{R}$, show that $nc_n \to 0$ as $n \to \infty$. [HINT: Consider the sum of the $(n+1)$st through $(2n)$th terms .]

*Solution:* Since the series is uniformly convergent, we have

$$\sum_{k=n+1}^{2n} c_k \sin kx \to 0, \text{ uniformly in } x \text{ as } n \to \infty.$$

But now choose $x \equiv x_n = \pi/(3n)$. Then $\sin kx_n \geq 1/2$ for $n+1 \leq k \leq 2n$. Therefore,

$$0 < \tfrac{n}{2} c_{2n} \leq \tfrac{1}{2} \sum_{k=n+1}^{2n} c_k \leq \sum_{k=n+1}^{2n} c_k \sin kx_n \to 0 \text{ as } n \to \infty,$$

which immediately yields the desired result.

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15. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous (but not necessarily differentiable) function without local minimum or maximum. Prove that $f$ is monotonic (either non-increasing over $\mathbb{R}$ or non-decreasing over $\mathbb{R}$).

(A function $f$ is said to have a local maximum (resp. minimum) at $x$ if there exists $\varepsilon > 0$ such that $|y - x| \leq \varepsilon \Rightarrow f(y) \leq f(x)$ (resp. $f(y) \geq f(x)$).)

*Solution:* Let's prove that for any $x < y \in \mathbb{R}$ with $f(x) \leq f(y)$ (resp. $f(y) \leq f(x)$) we have, for all $z \in [x,y]$, $f(x) \leq f(z) \leq f(y)$ (resp. $f(y) \leq f(z) \leq f(x)$).

It suffices to consider the first case, since the other one can be deduced from it by replacing $f$ by $-f$. Assume that there exists $z \in (x,y)$ such that $f(z) < f(x)$. Then $\min_{[x,y]} f(u) < f(x) \leq f(y)$. Since $f$ is continuous, we know that the minimum is attained at some $z_0 \in (x,y)$ but this $z_0$ would be a local minimum (taking $\varepsilon = \min(|x - z_0|, |y - z_0|)$), which is a contradiction. Similarly, there is no $z \in (x,y)$ such that $f(z) > f(y)$.

So, $f$ is monotonic over any bounded interval. Assume that $f$ is not constant and that there exists $x < y$ such that $f(x) < f(y)$ (resp. $f(y) < f(x)$). Then, $f$ must be non-decreasing (resp. non-increasing) over any interval containing $[x,y]$ and therefore over $\mathbb{R}$.

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