Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Spring Session

January 22, 2009

Instructions: Read carefully!

1. This **closed-book** examination consists of 15 problems, each worth 5 points. The passing grade has been set at 50 points, i.e. 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been grouped by topic, but there are roughly equally many mainly motivated by each of the three areas identified in the syllabus (linear algebra; real analysis; probability;). Nor have the problems been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. The examination will begin at 8:30 AM; lunch and refreshments will be provided. The exam will end just before 5:00 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Let A be a Hermitian, tridiagonal, square matrix of order n, with all its subdiagonals and superdiagonals nonzero (i.e., $A = (a_{ij}, i, j = 1, \ldots, n)$ is such that $a_{ij} = 0$ if $|i - j| > 1$ and $a_{ij} \neq 0$ if $|i - j| = 1$). Prove that the eigenvalues of A are distinct.

2. Let $A = \left(a_{ij}\right)_{i,j=1,\ldots,n}$ be the $n \times n$ matrix whose entries satisfy

$$a_{ij} = \begin{cases} 1 & \text{if } j \geq i \\ -1 & \text{if } j = i - 1 \\ 0 & \text{otherwise,} \end{cases}$$

so that, for example, if $n = 3$ the matrix takes the form

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

What is $\det(A)$? Justify your answer rigorously.

3. Let f be a k times differentiable real-valued function on a nonempty finite open interval (a, b), k being a positive integer. Show that if the k-th derivative $f^{(k)}$ is bounded on (a, b), then f is uniformly continuous on (a, b).

4. Let $f : [0, 1] \to [0, 1]$ be continuous. Suppose there is a value $a \in [0, 1]$ such that $f[f(a)] = a$. Prove there is a value $b \in [0, 1]$ (not necessarily different from a) such that $f(b) = b$.

5. Suppose that $\mathbf{h} = (\alpha, \beta, \gamma)$ is a smooth, 1-1 function from $V \subset \mathbb{R}^3$ onto $W \subset \mathbb{R}^3$, where V, W are open neighborhoods. Use the coordinate functions to define two vector fields on V :

$$\mathbf{f}(\mathbf{x}) = \nabla\alpha(\mathbf{x}) \times \nabla\beta(\mathbf{x}), \ \ \mathbf{g}(\mathbf{x}) = \nabla\gamma(\mathbf{x})$$

Prove that

$$\int_V \mathbf{f}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x})\, d\mathbf{x} = \pm \int_W d\mathbf{y}.$$

6. Let X, Y be independent random variables, each with density $f(x) = x^{-2}, x \geq 1$ and $f(x) = 0$ otherwise. Calculate the probability density function of the vector (U, V) where

$$U = XY, \ \ V = \frac{X}{Y}.$$

7. Let A and B be two matrices such that $B-A$, A^{-1} and B^{-1} all have positive coefficients. Show that $A^{-1}-B^{-1}$ also has positive coefficients.

8. Let X and Y be two random variables with respective probability density functions f_X and f_Y, f_X and f_Y being continuous functions defined on $\mathbb{R}$.

   Prove that, if $f_X(x) > 0 \Leftrightarrow f_Y(x) > 0$ for $x \in \mathbb{R}$, then, $P(X \in I) > 0 \Leftrightarrow P(Y \in I) > 0$ for any interval I in $\mathbb{R}$.

   Is the converse statement also true? Justify your answer.

9. Your two friends are to meet you at a particular time. The friends independently arrive late, by random amounts of time, X and Y, which are exponentially distributed with a mean of five minutes. What is the expected waiting time until the second friend arrives?

10. Let n be a fixed positive integer, and suppose that X and Θ are random variables such that the conditional distribution of X given $\Theta = p$ is a binomial with parameters n and p. Further assume that Θ has a beta distribution with parameters α and β ($\alpha, \beta > 0$), i.e., Θ has the following p.d.f.:

$$f_\Theta(p) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1} & \text{if } 0 < p < 1 \\ 0 & \text{elsewhere} \end{cases} .$$

    Determine the conditional distribution of Θ given $X = x$.

11. Let a, b, c, d be distinct real numbers. Consider the four points in $\mathbb{R}^3$:

$$A = \begin{bmatrix} a \\ a^2 \\ a^3 \end{bmatrix}, \quad B = \begin{bmatrix} b \\ b^2 \\ b^3 \end{bmatrix}, \quad C = \begin{bmatrix} c \\ c^2 \\ c^3 \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} d \\ d^2 \\ d^3 \end{bmatrix}.$$

    Prove that lines AB and CD do not intersect.

12. Let S and T be two subspaces in R^n and A be a $n \times n$ real matrix. Show that if $\dim(S) > \dim(T)$ and $Ax \in T$ for any $x \in S$ then there exists a nonzero $x \in S$ such that $Ax = 0$.

13. In a laboratory, two types of insects are generated every day. Each new insect is equally likely to be one of two possible types. Assume that, on any given day, the total number of

insects generated is Poisson distributed with parameter λ. For a certain day, let N_i denote the number of insects of type i that are generated ($i = 1,2$). What is the probability that $N_1 > 0$ and $N_2 = 0$? Express your answer as simply as possible in terms of λ.

[HINT: One solution strategy is to condition on the value of $N := N_1 + N_2$.]

14. Let (c_n) be a decreasing sequence of positive numbers. If the series $\sum(c_n \sin nx)$ is uniformly convergent for $x \in \mathbb{R}$, show that $nc_n \to 0$ as $n \to \infty$. [HINT: Consider the sum of the $(n+1)$st through $(2n)$th terms .]

15. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous (but not necessarily differentiable) function without local minimum or maximum. Prove that f is monotonic (either non-increasing over $\mathbb{R}$ or non-decreasing over $\mathbb{R}$).

    (A function f is said to have a local maximum (resp. minimum) at x if there exists $\varepsilon > 0$ such that $|y - x| \leq \varepsilon \Rightarrow f(y) \leq f(x)$ (resp. $f(y) \geq f(x)$).)