

Department of Applied Mathematics and Statistics
The Johns Hopkins University

INTRODUCTORY EXAMINATION—SPRING SESSION

January 23, 2008

Instructions: Read carefully!

1. This **closed-book** examination consists of 20 problems (sorry, no choices), each worth 5 points. The passing grade has been set at $66\frac{2}{3}\%$. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.
3. The problems have not been grouped by topic, but there are roughly equally many mainly motivated by each of the four areas identified in the syllabus (linear algebra; real analysis; probability; discrete mathematics and operations research/optimization). Nor have the problems been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
4. Start your answer to each problem on a **NEW** sheet of paper. Write only on **ONE SIDE** of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your **NAME** and the **PROBLEM NUMBER** on each sheet.
5. The examination will begin at 8:30 AM; lunch and refreshments will be provided. The exam will end just before 5:00 PM. You may leave before then, but in that case you may not return.
6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
7. **No calculators of any sort are needed or permitted.**

1. Alice and Bob play a game with four balanced dice. At each round of the game, both players roll two dice. If both of Alice's dice show larger numbers than both of Bob's, then Alice is the winner of the game. If both of Alice's dice show smaller numbers than both of Bob's, then Bob is the winner. Otherwise, another round is played, and the game continues until there is a winner.

What is the expected number of rounds played until there is a winner?

Solution

The student should assume that the dice rolls are independent and equally likely. The expected number of rounds played is then a Geometric random variable, so its expectation is the reciprocal of its parameter. The parameter is the probability that someone wins on a single round.

Calculate the probability that Alice wins on a single round, by conditioning on the maximum number rolled by Bob, denoted by M .

$$P[M = k] = \frac{2k - 1}{36}$$

and

$$P[\text{Alice wins} | M = k] = \frac{(6 - k)^2}{36}$$

so

$$\begin{aligned} P[\text{Alice wins}] &= \sum_{k=1}^5 \frac{2k - 1}{36} \frac{(6 - k)^2}{36} \\ &= \frac{(1)(25) + (3)(16) + (5)(9) + (7)(4) + (9)(1)}{1296} \\ &= \frac{155}{1296} \end{aligned}$$

Thus, the probability that Alice or Bob wins on a given round is $\frac{155}{648}$, so the expected number of rounds is $\frac{648}{155}$.

2. Use the power series expansion of the exponential function to prove that

$$e^{x+y} = e^x e^y$$

for all $x, y \in \mathbb{R}$.

Solution

$$\begin{aligned} e^{x+y} &= \sum_{k=0}^{\infty} (x+y)^k / k! = \sum_{k=0}^{\infty} \sum_{m=0}^k \binom{k}{m} x^m y^{k-m} / k! \\ &= \sum_{k=0}^{\infty} \sum_{m=0}^k \left\{ \binom{k}{m} / k! \right\} x^m y^{k-m} \\ &= \sum_{m=0}^{\infty} \sum_{k=m}^{\infty} \frac{1}{m!(k-m)!} x^m y^{k-m} \\ &= \sum_{m=0}^{\infty} x^m / m! \sum_{k=m}^{\infty} y^{k-m} / (k-m)! = \sum_{m=0}^{\infty} x^m / m! \sum_{k=0}^{\infty} y^k / k! = e^x e^y. \end{aligned}$$

-
3. If A and B are $n \times n$ real symmetric matrices, write $A \leq B$ if and only if $B - A$ is nonnegative definite.

Show that if A and B are $n \times n$ real symmetric matrices with $A \leq B$, then $CAC^T \leq CBC^T$ for any C .

Solution $B - A$ is nonnegative definite, and also symmetric. $B - A$ symmetric nonnegative definite implies $B - A = EE^T$. Then $CBC^T - CAC^T = C(B - A)C^T = CEE^T C^T = (CE)(CE)^T$. Thus $CBC^T - CAC^T$ is symmetric nonnegative definite, and we write $CAC^T \leq CBC^T$.

-
4. Of the 100,000 five-digit sequences from 00000 to 99999, how many do not have three consecutive digits all the same?

Solution

Let A, B, C be the set of 5-digit sequences whose first three/second three/last three digits are identical, respectively. We seek to calculate $10^5 - |A \cup B \cup C|$.

Apply inclusion-exclusion:

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= 10^3 + 10^3 + 10^3 - 10^2 - 10 - 10^2 + 10 \\ &= 3000 - 200 = 2800 \end{aligned}$$

and so there are $10^5 - 2800$ or 97,200 sequence that do not have three consecutive digits all the same.

5. Use duality to check if vector $x=(1,0,1,0)$ is an optimal solution of the following linear program:

$$\begin{array}{ll} \min & -x_1 + 2x_2 - x_3 - x_4 \\ \text{s.t.} & x_1 + x_2 - x_3 + 2x_4 \geq -2 \\ & x_1 + 2x_2 - x_3 + x_4 = 0 \\ & -x_1 - x_2 - x_3 - x_4 \geq -2 \\ & x_1, x_2, x_3 \geq 0 \quad x_4 \text{ unrestricted.} \end{array}$$

Solution

We first consider the dual problem and find a corresponding dual solution to $x = (1, 0, 1, 0)$.

$$\begin{array}{ll} \min & -2w_1 \quad -2w_3 \\ \text{s.t.} & w_1 + w_2 - w_3 \leq -1 \\ & w_1 + 2w_2 - w_3 \leq 2 \\ & -w_1 - w_2 - w_3 \leq -1 \\ & 2w_1 + w_2 - w_3 = -1 \\ & w_1, w_3 \geq 0 \quad w_2 \text{ unrestricted.} \end{array}$$

By $x_1 > 0$ and $x_3 > 0$, we have the first and the third constraints of the dual as equalities: $w_1 + w_2 - w_3 = -1$ and $-w_1 - w_2 - w_3 = -1$. The fourth constraint $2w_1 + w_2 - w_3 = -1$ is another equality. Solve the system to get the dual solution $w = (0, 0, 1)$. Then we check (1) x is feasible to the original problem, (2) w is feasible to the dual, (3) the two objective function values have the same value -2, therefore x is an optimal solution.

6. A number X is chosen uniformly from the set $\{1, \dots, n\}$. Find $E(X)$ and $\text{Var}(X)$.
-

Solution

The easiest way to find the expected value is to note the symmetry of the uniform distribution about $(n+1)/2$; thus $EX = (n+1)/2$. Next,

$$EX^2 = \frac{1}{n} \sum_{x=1}^n x^2 = \frac{(n+1)(2n+1)}{6}.$$

Therefore,

$$\begin{aligned}\text{Var}X &= EX^2 - (EX)^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = \frac{(n+1)[(4n+2) - (3n+3)]}{12} \\ &= \frac{(n+1)(n-1)}{12} = \frac{n^2-1}{12}.\end{aligned}$$

7. Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be the function defined by $f(x) := x|x|$. (In answering the following questions about f , you may find it notationally convenient to refer to the “signum” function sgn , which has value 1, -1 , 0 according as its argument is positive, negative, or zero.)

- (a) Show that f is differentiable and strictly increasing.
 - (b) By part (a), f must have a continuous and strictly increasing inverse function g . Obtain and justify a formula for $g(y)$.
-

Solution

- (a) On the positive x -axis, $f(x) = x^2$ so that $f'(x) = 2x > 0$; thus f is differentiable and (strictly) increasing there, and also positive. Similarly, on the negative x -axis, $f(x) = -x^2$ so that $f'(x) = -2x > 0$, implying that f is differentiable and increasing, as well as negative, there. Because $f(0) = 0$, while $\text{sgn}[f(x)] = \text{sgn}(x)$ for $x \neq 0$, it follows that f is increasing throughout \mathbf{R} .

It remains only to show that f is differentiable at 0. Indeed, we will show that $f'(0) = 0$. As $x \rightarrow 0$ we have

$$\frac{f(x) - f(0)}{x - 0} = \frac{x|x|}{x} = |x| \rightarrow 0,$$

giving the desired derivative.

- (b) Reversing the axes in the graph of f , and staring at the result, leads to the claim that $g(y) = \text{sgn}(y)\sqrt{|y|}$. Indeed, with this choice,

- for $x = 0$ we have $g(f(0)) = g(0) = 0 = x$,
- for $x > 0$ we have $g(f(x)) = \text{sgn}(x^2)\sqrt{|x^2|} = 1 \times x = x$, and finally
- for $x < 0$ we have $g(f(x)) = \text{sgn}(-x^2)\sqrt{|-x^2|} = (-1)\sqrt{x^2} = (-1)(-x) = x$.

So $g(f(x)) = x$ for all x , showing that g is the inverse of f .

8. Let A be an $n \times n$ matrix with the properties:

- $A^2 \neq 0$, but
- $A^3 = 0$.

Prove that $n \geq 3$.

Solution

Trivially, $n \neq 1$, so suppose (for contradiction) $n = 2$. Since every matrix is similar to an upper triangular matrix, wolog A is upper triangle. The eigenvalues of A must be zero for if v is an eigenvector with eigenvalue λ , then $0 = A^2v = \lambda^2v$ which implies $\lambda = 0$. Since the eigenvalues of an upper triangular matrix are on the diagonal, we must have $A = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$ where $a \neq 0$. But then $A^2 = 0$, a contradiction. Thus $n \geq 3$.

COMMENT: Although this is short, I think it might be a bit on the difficult side. Is there an easier proof? Or should we give a hint such as: Recall that every matrix is similar to an upper triangular matrix. But then, I fear, the problem becomes too easy since its not hard to show that

$$\begin{bmatrix} a & * \\ 0 & b \end{bmatrix}^3 = \begin{bmatrix} a^3 & * \\ 0 & b^3 \end{bmatrix}$$

forcing $a = b = 0$ (which skips the eigenvalue approach).

Alternate Solution: Since $A^3 = 0$, the minimal polynomial $m(x)$ divides x^3 , so $m(x) = x^k$ for some k . Since A^2 and hence A are nonzero, the minimal polynomial must be x^3 . However, $m(x)$ must divide the characteristic polynomial $p(x) = \det(A - xI)$ which is of degree n , (the dimension of A) so it must be the case that $n \geq 3$.

9. Let A be a real $n \times n$ symmetric matrix. Prove that if c is sufficiently large, then $A + cI$ is positive definite.

Solution

Clearly $A + cI$ is a real symmetric matrix so we need only show that for sufficiently large c we have

$$v^t(A + cI)v \geq 0.$$

for all n -vectors v . By the spectral theorem, we can write $PAP^t = D$ where P is $n \times n$ orthogonal and $D = \text{diag}(d_1, \dots, d_n)$ is diagonal. Then we can write

$$v^t(A + cI)v = (Pv)^t(P(A + cI)P^t)(Pv) = (Pv)^t(D + cI)(Pv) = \sum_{i=1}^n (d_i + c)(Pv)_i^2,$$

which is nonnegative as long as $c \geq -\min_i d_i$.

10. Let X_1 and X_2 be independent random variables, each uniformly distributed on $\{1, 2, \dots, N\}$. Express the mean of $Y = \max\{X_1, X_2\}$ as a function of N .
-

Solution

First,

$$P(Y \leq n) = P(X_1 \leq n, X_2 \leq n) = \left(\frac{n}{N}\right)^2, \quad 1 \leq n \leq N.$$

As a result,

$$P(Y = n) = \left(\frac{n}{N}\right)^2 - \left(\frac{n-1}{N}\right)^2 = \frac{2n-1}{N^2}, \quad 1 \leq n \leq N.$$

Therefore

$$\begin{aligned} E(Y) &= \sum_{n=1}^N nP(Y = n) \\ &= \sum_{n=1}^N n \left(\frac{2n-1}{N^2} \right). \end{aligned}$$

This can easily be evaluated:

$$\begin{aligned} &= \frac{1}{N^2} \left[2 \sum_{n=1}^N n^2 - \sum_{n=1}^N n \right] \\ &= \frac{1}{N^2} \left[2 \frac{N}{6} (N+1)(2N+1) - \frac{N(N+1)}{2} \right] \\ &= \frac{(N+1)(2N+1)}{3N} - \frac{N+1}{2N}. \end{aligned}$$

-
11. Let S be a set of $n \geq 2$ distinct points in the plane. We want to find a closed circular disk of minimum size that contains all these points.

Express this problem as a convex program. (Show explicitly why your formulation meets the definition of “convex program”.)

Solution

Our mathematical program will have three variables: the radius r of the disk and the two coordinates of x , its center. Denote the points in S by x_1, \dots, x_n . Then the problem reads:

$$\text{Minimize } r \text{ subject to } |x - x_j| \leq r \text{ for all } j,$$

where $|v|$ denotes the Euclidean length of a vector v in \mathbf{R}^2 . The objective function (r) is linear, hence convex, in the variables. Since an intersection of convex sets is convex, the remaining condition for a convex program can be verified by showing that each of the n constraints defines a convex set in \mathbf{R}^3 . Indeed, if each of (x', r') and (x'', r'') satisfies $|x - x_j| \leq r$, then for $0 \leq t \leq 1$ the point (y, s) with

$$y = (1-t)x' + tx'' \quad \text{and} \quad s = (1-t)r' + tr''$$

also satisfies it, because

$$\begin{aligned} |y - x_j| &= |(1-t)(x' - x_j) + t(x'' - x_j)| \leq |(1-t)(x' - x_j)| + |t(x'' - x_j)| \\ &= (1-t)|x' - x_j| + t|x'' - x_j| \leq (1-t)r' + tr'' = s. \end{aligned}$$

-
12. Let X_1 and X_2 be two independent random variables such that $X_i \sim \Gamma(\alpha_i, 1)$, $i = 1, 2$, with $\alpha_1, \alpha_2 > 0$.

Let $Y_1 = X_1/(X_1 + X_2)$ and $Y_2 = X_1 + X_2$. Prove that Y_1 and Y_2 are independent and identify their distributions.

Solution

The joint density of (X_1, X_2) is

$$\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} x_1^{\alpha_1-1} x_2^{\alpha_2-1} e^{-x_1-x_2}$$

on $(0, +\infty)^2$. The transformation $(x_1, x_2) \rightarrow (x_1/(x_1 + x_2), x_1 + x_2)$ is invertible between and one passes from (y_1, y_2) to (x_1, x_2) with $(0, +\infty)^2$ and $(0, 1) \times (0, +\infty)$, with inverse $(y_1, y_2) \mapsto (y_1 y_2, y_2 - y_1 y_2)$. The Jacobian determinant of the inverse is

$$y_2(1 - y_1) - y_1(-y_2) = y_2$$

and the joint density distribution of (Y_1, Y_2) is

$$\begin{aligned} \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} (y_1 y_2)^{\alpha_1-1} (y_2(1 - y_1))^{\alpha_2-1} e^{-y_2} y_2 \\ = \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} y_1^{\alpha_1-1} (1 - y_1)^{\alpha_2-1} \times \frac{y_2^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1 + \alpha_2)} e^{-y_2} \end{aligned}$$

This proves that Y_1 and Y_2 are independent, with distributions $\beta(\alpha_1, \alpha_2, 1)$ and $\Gamma(\alpha_1 + \alpha_2, 1)$.

13. In one version of “Keno”, a player chooses 7 (distinct) numbers from the set $F = \{1, 2, \dots, 80\}$ and a machine chooses 20 (distinct) numbers from F . In both cases, all subsets are equally likely to be chosen. The player wins if all seven of the chosen numbers are among the twenty chosen by the machine. What is the probability the player wins?
-

Solution

We can regard the seven numbers as fixed and the randomness due to choice of the machine. Therefore,

$$P(\text{win}) = \frac{\binom{73}{13}}{\binom{80}{20}}.$$

We could also suppose the machine chooses first, so twenty numbers are set aside. Then:

$$P(\text{win}) = \frac{\binom{20}{7}}{\binom{80}{7}}.$$

These are in fact the same.

14. Let f, g be continuous real-valued functions on \mathbb{R}^n , with g nonnegative, supported in the unit ball at the origin, and $\int_{\mathbb{R}^n} d\mathbf{x} g(\mathbf{x}) = 1$. For any $\delta > 0$, define $g_\delta(\mathbf{x}) = \delta^{-n} g(\mathbf{x}/\delta)$ and

$$\bar{f}_\delta(\mathbf{x}) = \int_{\mathbb{R}^n} d\mathbf{y} g_\delta(\mathbf{y}) f(\mathbf{x} + \mathbf{y}).$$

Prove that

$$\lim_{\delta \rightarrow 0} \bar{f}_\delta(\mathbf{x}) = f(\mathbf{x})$$

uniformly on compact subsets of \mathbb{R}^n .

Solution

Note that

$$\bar{f}_\delta(\mathbf{x}) - f(\mathbf{x}) = \int_{\mathbb{R}^n} d\mathbf{y} g_\delta(\mathbf{y}) [f(\mathbf{x} + \mathbf{y}) - f(\mathbf{x})],$$

and that g_δ is continuous, nonnegative, with unit integral, and supported in the ball of radius δ at the origin. Since f is continuous, it is uniformly continuous on compact sets K . Thus, for $\varepsilon_0 > 0$, there exists a $\delta_0 > 0$, such that $|\mathbf{y}| < \delta_0$ implies that $\sup_{\mathbf{x} \in K} |f(\mathbf{x} + \mathbf{y}) - f(\mathbf{x})| < \varepsilon_0$. Choosing $\delta < \delta_0$ and using the support of g_δ inside the ball of radius δ gives

$$\sup_{\mathbf{x} \in K} |\bar{f}_\delta(\mathbf{x}) - f(\mathbf{x})| \leq \int_{\mathbb{R}^n} d\mathbf{y} g_\delta(\mathbf{y}) \sup_{\mathbf{x} \in K} |f(\mathbf{x} + \mathbf{y}) - f(\mathbf{x})| < \varepsilon_0 \int_{\mathbb{R}^n} d\mathbf{y} g_\delta(\mathbf{y}) = \varepsilon_0.$$

Thus, $\bar{f}_\delta(\mathbf{x}) \rightarrow f(\mathbf{x})$ uniformly on K as $\delta \rightarrow 0$.

15. Let $\{a_1, \dots, a_n\}$ be a basis for a vector space V and $n \geq 2$. Is the set

$$\{a_1 + a_2, a_2 + a_3, \dots, a_{n-1} + a_n, a_n + a_1\}$$

also a basis for V ?

Solution

We only need to show that the second set is linearly independent. Consider

$$\lambda_1(a_1 + a_2) + \dots + \lambda_n(a_n + a_1) = 0.$$

Equivalently, we have

$$(\lambda_1 + \lambda_n)a_1 + (\lambda_1 + \lambda_2)a_2 + \dots + (\lambda_{n-1} + \lambda_n)a_n = 0.$$

It follows from the linear independence of $\{a_i\}$ that $\lambda_1 + \lambda_n = 0, \dots, \lambda_{n-1} + \lambda_n = 0$. This system has nontrivial solution if and only if

$$\det \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 & 1 \end{pmatrix} = 1 + (-1)^{n+1}$$

is equal to zero. Therefore, the set is a basis for V only when n is odd.

Alternate Solution: Here's another solution to this problem that does not use determinants.

Suppose

$$c_1(a_1 + a_2) + c_2(a_2 + a_3) + \dots + c_{n-1}(a_{n-1} + a_n) + c_n(a_n + a_1) = 0.$$

Then we have

$$(c_1 + c_n)a_1 + (c_1 + c_2)a_2 + (c_2 + c_3)a_3 + \dots + (c_{n-1} + c_n)a_n = 0.$$

By linear independence we conclude that the coefficients $c_1 + c_n, c_1 + c_2, c_2 + c_3, \dots, c_{n-1} + c_n$ are all zero. Thus, $c_2 = -c_1, c_3 = -c_2 = c_1, c_4 = -c_3 = c_2 = -c_1 \dots c_n = (-1)^{(n+1)}c_1$. and we have an additional equation $c_n = -c_1$.

If n is even then this system has a nontrivial solution e.g. $c_i = (-1)^i, i = 1, \dots, n$. However, if n is odd then the last two equations give $c_n = c_1 = -c_1$ so $c_n = 0$. Then it follows that all $c_i = 0$. This, we get linear independence of

$$a_1 + a_2, a_2 + a_3, \dots, a_{n-1} + a_n, a_n + a_1$$

if and only if n is odd.

16. If Z_1, \dots, Z_n are iid Normal(0,1) random variables, the distribution of $\sum_{i=1}^n Z_i^2$ is called the χ^2 distribution with n degrees of freedom. Suppose X has a χ^2 distribution with 50 degrees of freedom. Compute an approximation to $\Phi^{-1}(P[X \leq 70])$ where Φ is the standard normal cumulative distribution function.

Solution

We have that X is distributed as $\sum_{i=1}^{50} Z_i^2$. We get the mean and variance of each term Z_i^2 as follows:

$$E[Z_i^2] = \text{Var}(Z_i) = 1$$

$$\text{Var}(Z_i^2) = E[Z_i^4] - E[Z_i^2]^2 = E[Z_i^4] - 1.$$

In addition,

$$E[Z_i^4] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^4 e^{-\frac{1}{2}z^2} dz$$

and integration by parts with $u = z^3$ and $dv = ze^{-\frac{1}{2}z^2}$ gives

$$E[Z_i^4] = -\frac{1}{\sqrt{2\pi}} z^3 e^{-\frac{1}{2}z^2} \Big|_{z=-\infty}^{\infty} + \frac{1}{\sqrt{2\pi}} \int_{z=-\infty}^{\infty} 3z^2 e^{-\frac{1}{2}z^2} dz = 3.$$

So $\text{Var}(Z_i^2) = 3 - 1 = 2$.

The central limit theorem gives

$$\frac{X - n}{\sqrt{2n}} \xrightarrow{d} N(0, 1)$$

so we have

$$P\left[\frac{X - 50}{\sqrt{100}} \leq t\right] = P[X \leq 50 + 10t] \approx \Phi(t).$$

Thus

$$\Phi^{-1}(P[X \leq 50 + 10t]) \approx t.$$

Taking $t = 2$ gives

$$\Phi^{-1}(P[X \leq 70]) \approx 2.$$

17. A continuous curve is a continuous function $f : [0, 1] \rightarrow \mathbb{R}^2$. An inscribed polygonal line in f is a polygonal line with vertices $(t_i, f(t_i))$ for some finite family (t_1, \dots, t_n) with $0 \leq t_1 < \dots < t_n \leq 1$ (n may vary).

Define $L(f) = \sup L(q)$, the (possibly infinite) supremum being taken over all polygonal lines inscribed in f (the length of a polygonal curve is the sum of lengths of its edges).

Prove that, for $\alpha \in \mathbb{R}$, and two continuous curves f_1 and f_2 , $L(\alpha f_1) = |\alpha|L(f_1)$ and $L(f_1 + f_2) \leq L(f_1) + L(f_2)$.

Solution

The statement is obvious when $\alpha = 0$, so assume that $\alpha \neq 0$. If q is an inscribed polygonal line in f , then αq is a polygonal line in αf and $L(\alpha q) = |\alpha|L(q)$ so that

$$|\alpha|L(q) \leq L(\alpha f)$$

and taking the supremum of the right hand side, $|\alpha|L(f) \leq L(\alpha f)$. But applying this with $1/\alpha$ and αf yields

$$|\alpha|^{-1}L(\alpha f) \leq L(f)$$

which proves $|\alpha|L(f) = L(\alpha f)$.

For the sum, let $q(t_k) = f_1(t_k) + f_2(t_k), k = 1, \dots, n$ be a polygonal line inscribed in $f_1 + f_2$. Then letting $q_i(t_k) = f_i(t_k)$ for $k = 1, \dots, n$ and $i = 1, 2$, q_1 and q_2 are polygonal line respectively inscribed in f_1 and f_2 so that $L(q_1) + L(q_2) \leq L(f_1) + L(f_2)$. But, since

$$|q(t_k) - q(t_{k-1})| \leq |q_1(t_k) - q_1(t_{k-1})| + |q_2(t_k) - q_2(t_{k-1})|$$

we have $L(q) \leq L(q_1) + L(q_2) \leq L(f_1) + L(f_2)$ and taking the supremum over q yields the result.

18. Let

$$f(x) := \lim_n (\cos x)^{2n}.$$

- (a) For all real numbers x , show that this sequential limit exists and evaluate $f(x)$.
- (b) Is the function f Riemann integrable over $[0, 10]$? Give a proof; and if your answer is yes, what is the value of $\int_0^{10} f(x) dx$?

Solution

- (a) If x is not an integer multiple of π , then $|\cos x| < 1$ and so $f(x) = 0$. If x is an integer multiple of π , then $|\cos x| = 1$ and so $f(x) = 1$.
- (b) The function f is piecewise continuous over $[0, 10]$ and so is Riemann integrable. The value of the Riemann integral of a piecewise continuous function f defined on a compact interval does not depend on the values of f at the points of discontinuity, so f has the same integral as the zero-function, namely, 0.

19. For any set of n vectors $\tilde{\zeta}_1, \dots, \tilde{\zeta}_n$ in a finite-dimensional inner product space over \mathbb{C} , define the *Gramian matrix* $\mathbf{G} = \mathbf{G}[\tilde{\zeta}_1, \dots, \tilde{\zeta}_n]$ to have the matrix elements

$$G_{ij} = (\tilde{\zeta}_i, \tilde{\zeta}_j).$$

- (a) Show that every Gramian matrix is Hermitian, non-negative-definite.
 (b) Now show the converse: every Hermitian, non-negative-definite $n \times n$ matrix is the Gramian matrix for some set of n vectors $\tilde{\zeta}_1, \dots, \tilde{\zeta}_n$ in a finite-dimensional inner product space.

Solution

(a) Since $\overline{(\tilde{\zeta}_i, \tilde{\zeta}_j)} = (\tilde{\zeta}_j, \tilde{\zeta}_i)$ is one of the defining properties of an inner product, \mathbf{G} is Hermitian. Furthermore,

$$\sum_{ij} c_i \bar{c}_j (\tilde{\zeta}_i, \tilde{\zeta}_j) = \left\| \sum_i c_i \tilde{\zeta}_i \right\|^2 \geq 0,$$

so that \mathbf{G} is also non-negative-definite.

(b) By the spectral theorem,

$$\mathbf{G} = \sum_{\alpha=1}^n \lambda_{\alpha} \mathbf{e}_{\alpha} \mathbf{e}_{\alpha}^*,$$

where λ_{α} are the eigenvalues and \mathbf{e}_{α} the eigenvectors of the $n \times n$ Hermitian matrix. Since \mathbf{G} is non-negative, all the eigenvalues are non-negative. Thus,

$$G_{ij} = \sum_{\alpha=1}^n v_{i,\alpha} \bar{v}_{j,\alpha} = (\tilde{\zeta}_i, \tilde{\zeta}_j),$$

where \mathbf{v}_i is the element of \mathbb{C}^n with components $v_{i,\alpha} = (\mathbf{v}_i)_{\alpha} = \sqrt{\lambda_{\alpha}} (\mathbf{e}_{\alpha})_i$ and $(\tilde{\zeta}_i, \tilde{\zeta}_j)$ is the standard Euclidean inner product on \mathbb{C}^n .

20. Let G be a planar graph in which each vertex has degree 3. Show that, if G has a planar representation in which each face is bounded by precisely four or six edges, then there are exactly six faces bounded by four edges.

Hint: Let x be the number of faces bounded by four edges and y be the number of faces bounded by six edges. Using graph relationships, write a set of equations in x and y , which has a unique solution for x but not for y !

Solution

Fact 1. By Euler's formula,

$$|V| - |E| + x + y = 2,$$

or equivalently

$$x + y = |E| - |V| + 2.$$

Fact 2. By adding the numbers of edges around all the faces,

$$2|E| = 4x + 6y,$$

giving

$$|E| = 3(x + y) - x.$$

Fact 3. By adding degrees of all the vertices,

$$2|E| = 3|V|.$$

Multiplying the first fact by 3, we obtain

$$3|E| - 3|V| + 6 = 3(x + y),$$

which, from the third fact, becomes

$$|E| = 3(x + y) - 6.$$

Comparison with the second fact gives the solution $x = 6$.