Department of Applied Mathematics and Statistics
The Johns Hopkins University

INTRODUCTORY EXAMINATION–SPRING SESSION

January 23, 2008

Instructions: Read carefully!

1. This **closed-book** examination consists of 20 problems (sorry, no choices), each worth 5 points. The passing grade has been set at $66\frac{2}{3}\%$. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.
3. The problems have not been grouped by topic, but there are roughly equally many mainly motivated by each of the four areas identified in the syllabus (linear algebra; real analysis; probability; discrete mathematics and operations research/optimization). Nor have the problems been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.
5. The examination will begin at 8:30 AM; lunch and refreshments will be provided. The exam will end just before 5:00 PM. You may leave before then, but in that case you may not return.
6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
7. **No calculators of any sort are needed or permitted.**

---

1. Alice and Bob play a game with four balanced dice. At each round of the game, both players roll two dice. If both of Alice's dice show larger numbers than both of Bob's, then Alice is the winner of the game. If both of Alice's dice show smaller numbers than both of Bob's, then Bob is the winner. Otherwise, another round is played, and the game continues until there is a winner.

   What is the expected number of rounds played until there is a winner?

---

2. Use the power series expansion of the exponential function to prove that
   $$e^{x+y} = e^x e^y$$
   for all $x, y \in \mathbb{R}$.

---

3. If A and B are $n \times n$ real symmetric matrices, write $A \leq B$ if and only if $B - A$ is nonnegative definite.
   Show that if A and B are $n \times n$ real symmetric matrices with $A \leq B$, then $CAC^T \leq CBC^T$ for any C.

---

4. Of the 100,000 five-digit sequences from 00000 to 99999, how many do not have three consecutive digits all the same?

---

5. Use duality to check if vector x=(1,0,1,0) is an optimal solution of the following linear program:
   $$\begin{array}{lll} \text{min} & -x_1 + 2x_2 - x_3 - x_4 & \\ \text{s.t.} & x_1 + x_2 - x_3 + 2x_4 & \geq -2 \\ & x_1 + 2x_2 - x_3 + x_4 & = 0 \\ & -x_1 - x_2 - x_3 - x_4 & \geq -2 \\ & x_1, x_2, x_3 \geq 0 & x_4 \quad \text{unrestricted.} \end{array}$$

---

6. A number X is chosen uniformly from the set $\{1,\dots,n\}$. Find $E(X)$ and $\mathrm{Var}(X)$.

---

7. Let $f : \mathbf{R} \to \mathbf{R}$ be the function defined by $f(x) := x|x|$. (In answering the following questions about f, you may find it notationally convenient to refer to the "signum" function sgn, which has value $1, -1, 0$ according as its argument is positive, negative, or zero.)

   (a) Show that f is differentiable and strictly increasing.

   (b) By part (a), f must have a continuous and strictly increasing inverse function g. Obtain and justify a formula for $g(y)$.

---

8. Let A be an $n \times n$ matrix with the properties:

   - $A^2 \neq 0$, but
   - $A^3 = 0$.

   Prove that $n \geq 3$.

---

9. Let A be a real $n \times n$ symmetric matrix. Prove that if c is sufficiently large, then $A + cI$ is positive definite.

---

10. Let X_1 and X_2 be independent random variables, each uniformly distributed on $\{1, 2, ..., N\}$. Express the mean of $Y = \max\{X_1, X_2\}$ as a function of N.

---

11. Let S be a set of $n \geq 2$ distinct points in the plane. We want to find a closed circular disk of minimum size that contains all these points.

    Express this problem as a convex program. (Show explicitly why your formulation meets the definition of "convex program".)

---

12. Let X_1 and X_2 be two independent random variables such that $X_i \sim \Gamma(\alpha_i, 1)$, $i = 1, 2$, with $\alpha_1, \alpha_2 > 0$.

    Let $Y_1 = X_1/(X_1 + X_2)$ and $Y_2 = X_1 + X_2$. Prove that Y_1 and Y_2 are independent and identify their distributions.

---

13. In one version of "Keno", a player chooses 7 (distinct) numbers from the set $F = \{1, 2, ..., 80\}$ and a machine chooses 20 (distinct) numbers from F. In both cases, all subsets are equally likely to be chosen. The player wins if all seven of the chosen numbers are among the twenty chosen by the machine. What is the probability the player wins?

---

14. Let f, g be continuous real-valued functions on $\mathbb{R}^n$, with g nonnegative, supported in the unit ball at the origin, and $\int_{\mathbb{R}^n} d\mathbf{x}\, g(\mathbf{x}) = 1$. For any $\delta > 0$, define $g_\delta(\mathbf{x}) = \delta^{-n} g(\mathbf{x}/\delta)$ and

    $$\bar{f}_\delta(\mathbf{x}) = \int_{\mathbb{R}^n} d\mathbf{y}\, g_\delta(\mathbf{y}) f(\mathbf{x} + \mathbf{y}).$$

    Prove that

    $$\lim_{\delta \to 0} \bar{f}_\delta(\mathbf{x}) = f(\mathbf{x})$$

    uniformly on compact subsets of $\mathbb{R}^n$.

---

15. Let $\{a_1, \cdots, a_n\}$ be a basis for a vector space V and $n \geq 2$. Is the set

    $$\{a_1 + a_2,\ a_2 + a_3, \cdots,\ a_{n-1} + a_n,\ a_n + a_1\}$$

    also a basis for V?

---

16. If $Z_1, \ldots, Z_n$ are iid Normal(0,1) random variables, the distribution of $\sum_{i=1}^n Z_i^2$ is called the χ^2 distribution with n degrees of freedom. Suppose X has a χ^2 distribution with 50 degrees of freedom. Compute an approximation to $\Phi^{-1}(P[X \leq 70])$ where Φ is the standard normal cumulative distribution function.

---

17. A continuous curve is a continuous function $f : [0,1] \to \mathbb{R}^2$. An inscribed polygonal line in f is a polygonal line with vertices $(t_i, f(t_i))$ for some finite family $(t_1, \ldots, t_n)$ with $0 \le t_1 < \cdots < t_n \le 1$ (n may vary).

    Define $L(f) = \sup L(q)$, the (possibly infinite) supremum being taken over all polygonal lines inscribed in f (the length of a polygonal curve is the sum of lengths of its edges).

    Prove that, for $\alpha \in \mathbb{R}$, and two continuous curves f_1 and f_2, $L(\alpha f_1) = |\alpha| L(f_1)$ and $L(f_1 + f_2) \le L(f_1) + L(f_2)$.

---

18. Let

$$f(x) := \lim_n (\cos x)^{2n}.$$

    (a) For all real numbers x, show that this sequential limit exists and evaluate $f(x)$.

    (b) Is the function f Riemann integrable over $[0,10]$? Give a proof; and if your answer is yes, what is the value of $\int_0^{10} f(x)\,dx$?

---

19. For any set of n vectors $\succapprox_1, \ldots, \succapprox_n$ in a finite-dimensional inner product space over $\mathbb{C}$, define the *Gramian matrix* $\mathbf{G} = \mathbf{G}[\succapprox_1, \ldots, \succapprox_n]$ to have the matrix elements

$$G_{ij} = (\succapprox_i, \succapprox_j).$$

    (a) Show that every Gramian matrix is Hermitian, non-negative-definite.

    (b) Now show the converse: every Hermitian, non-negative-definite $n \times n$ matrix is the Gramian matrix for some set of n vectors $\succapprox_1, \ldots, \succapprox_n$ in a finite-dimensional inner product space.

---

20. Let G be a planar graph in which each vertex has degree 3. Show that, if G has a planar representation in which each face is bounded by precisely four or six edges, then there are exactly six faces bounded by four edges.

    Hint: Let x be the number of faces bounded by four edges and y be the number of faces bounded by six edges. Using graph relationships, write a set of equations in x and y, which has a unique solution for x but not for y!