Department of Applied Mathematics and Statistics
The Johns Hopkins University

INTRODUCTORY EXAMINATION–SPRING SESSION

Tuesday, January 24, 2006

Instructions: Read carefully!

1. This **closed-book** examination consists of 20 problems (sorry, no choices), each worth 5 points. The passing grade has been set at $66\frac{2}{3}\%$. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been grouped by topic, but there are roughly equally many mainly motivated by each of the four areas identified in the syllabus (linear algebra; real analysis; probability; discrete mathematics and operations research/optimization). Nor have the problems been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. The examination will begin at 8:30 AM; lunch and refreshments will be provided. The exam will end just before 5:00 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Let $S = \{v_1, v_2, v_3\}$ be a basis for $\mathbb{R}^3$, where

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}, \quad \text{and} \quad v_3 = \begin{bmatrix} 2 \\ 9 \\ 0 \end{bmatrix}.$$

(a) Find the coordinate vector of $v = \left[\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right]$ with respect to S.

(b) Find the vector v in $\mathbb{R}^3$ whose coordinate vector with respect to S is $[v]_S = \left[\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right]$.

2. Estimate the probability that in 60 independent tosses of a pair of fair dice the sum is never equal to four. Use the Poisson approximation to the Binomial.

3. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and let $a < b$ be reals. Suppose that $fg' - gf'$ never vanishes on the interval $[a, b]$ and that $f(a) = f(b) = 0$. Prove that g has a root in the interval $[a, b]$.

4. Let J be a 100×100 matrix all of whose entries are equal to 1. Find (with justification) the rank, nullity (dimension of the null space), determinant, and eigenvalues (with multiplicities) of J.

5. Let H be an $n \times n$ matrix and let I be the $n \times n$ identity matrix. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) = \log|\det(I + tH)|$$

Show that (a) f is differentiable at 0 and that (b) $f'(0) = \text{trace}(H)$.

6. Let c, b be real vectors and let A be a real matrix. Let $P(c, A, b)$ denote the linear program

$$\text{find } x \text{ to} \quad \max c^T x \quad \text{subject to} \quad Ax \leq b,\ x \geq 0.$$

Let $\mathscr{P}$ denote the set of all linear programs that can be expressed in the form $P(\_, \_, \_)$.

Let $L \in \mathscr{P}$. Although one typically considers the dual $D(L)$ as a minimization problem show that, in fact, $D(L) \in \mathscr{P}$ and that $D(D(L)) = L$.

7. Let $A = [a_{ij}]$ be an $n \times n$ matrix. Define c_k (where $1 \leq k \leq n$) to be the sum of the determinants of all the $k \times k$ principal submatrices of A (a principal submatrix is a submatrix formed by taking any k rows and the same k columns). In this problem we ask you to prove that

$$|c_k| \leq \binom{n}{k} k^{k/2} \alpha^k$$

where $\alpha = \max_{ij} |a_{ij}|$.

To this end, you may use (without proof) Hadamard's inequality that

$$|\det B| = \prod_{i=1}^{m} \left(\sum_{j=1}^{m} |b_{ij}|^2 \right)^{1/2}$$

where $B = [b_{ij}]$ is any $m \times m$ matrix.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) := x|x|$. (In answering the following questions about f, you may find it notationally convenient to refer to the "signum" function sgn, which has value $1, -1, 0$ according as its argument is positive, negative, or zero.)

   (a) Show that f is differentiable and strictly increasing.

   (b) By part (a), f must have a continuous and strictly increasing inverse function g. Obtain and justify a formula for $g(y)$.

9. Let G be a (finite, simple) graph, $\Delta(G)$ the maximum degree in G, and $\chi(G)$ the chromatic number of G. Prove that $\chi(G) \leq \Delta(G) + 1$.

10. Let N be a random variable whose distribution is Poisson with parameter λ, so that

$$P[N = n] = e^{-\lambda} \lambda^n / n!, \; n = 0, 1, 2, \ldots,$$

and suppose that $X_1, X_2, \ldots,$ is a sequence of iid positive-valued random variables independent of N and whose expected value is μ. Show that

$$E\left[\prod_{i=1}^{N} X_i\right] = e^{\lambda(\mu - 1)}.$$

11. Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

(a) f is continuous at exactly one point x_0. What is the value of x_0 and prove that f is continuous there. (You do not need to show that f is not continuous elsewhere.)

(b) Is f differentiable at x_0? (Prove your answer.)

12. What is the expected number of times a fair die is rolled until all 6 sides appear at least once?

13. Prove that if K is a real skew-symmetric matrix and $I+K$ is nonsingular, then $(I-K)(I+K)^{-1}$ is orthogonal.

Note: K is skew-symmetric means that $K^T = -K$.

14. Two random variables X and Y have the joint density function

$$f_{(X,Y)}(x,y) = \frac{e^{-\frac{x^2}{2}+x}}{\sqrt{2\pi}} e^{-y} \text{ for } x \leq y \text{ and 0 otherwise.}$$

(a) Compute and identify the marginal density function of X.

(b) What is the conditional density function of Y given $X = x$?

15. Let $y, x_1, \ldots, x_n$ be (scalar) variables, and consider the single constraint

$$ny \geq x_1 + x_2 + \cdots + x_n \qquad (1)$$

and the family of constraints

$$y \geq x_j \text{ for all } j. \qquad (2)$$

(a) Show that if all the variables are binary (0 and 1 are the only values), then (**??**) and (**??**) are equivalent.

(b) In contrast, show that if all the variables are continuous with values in $[0,1]$, and also $n > 1$, then one of (**??**) and (**??**) (which one?) is strictly more restrictive than the other.

16. A man claims to have extrasensory perception (ESP): a supernatural ability to predict the future. As a test, before tossing a fair coin 10 times, he is asked to predict the outcomes in advance. The experiment is performed, and he gets 7 of 10 correct. What is the probability that he would have done at least this well if he does not have ESP?

17. Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation with the property that its matrix is the same relative to any basis. Show that this matrix must be a scalar multiple of the identity matrix.

    Hint: If there exists x for which $T(x)$ is not a scalar multiple of x, consider what the matrix of T looks like relative to a basis containing both x and $T(x)$.

18. Suppose that $x = (3,1)$ and $w = (0,2/3,1/3)$ are optimal solutions to the following problem and its dual, respectively.

$$\begin{array}{lrcl} \max & \alpha x_1 - x_2 & & \\ \text{s.t.} & -x_1 + x_2 & \leq & 2 \\ & x_1 + \beta x_2 & \leq & 1 \\ & \gamma x_1 + x_2 & \leq & 4 \\ & x_1, \quad x_2 & \geq & 0 \end{array}$$

    Find the values of α, β and γ.

19. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with the property that

$$f(x+y) = f(x) + f(y)$$

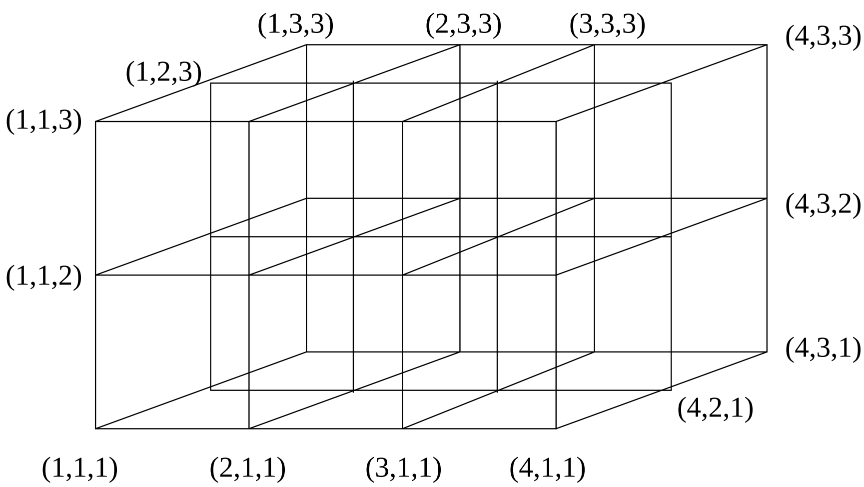
    for all values of $x, y \in \mathbb{R}$.

    Prove that $f(cx) = cf(x)$ for all $c, x \in \mathbb{R}$.

20. For given positive integers p, q and r, the $p \times q \times r$ wire mesh in $\mathbb{R}^3$ involves wires connecting pairs of points in the set

$$\mathscr{P}_{p,q,r} = \{(i,j,k) \;:\; 1 \le i \le p, 1 \le j \le q, 1 \le k \le r\},$$

and wire connections between every pair of points in $\mathscr{P}_{p,q,r}$ of the form $((i,j,k),(i+1,j,k))$, $((i,j,k),(i,j+1,k))$ or $((i,j,k),(i,j,k+1))$. For example, the figure below shows a $4 \times 3 \times 3$ mesh. Suppose an ant starts at the point $(1,1,1)$ and moves to the point (p,q,r) in the



$p \times q \times r$ wire mesh, using exactly $p+q+r-3$ steps, in such a way that

- at the end of every step the ant is still at a point with integer coordinates,
- each step involves moving a Euclidean distance of one unit.

How many distinct paths are there that the ant can follow?