Solutions to Introductory Exam for January, 2005

1. The function $f : [0,1] \to \mathbb{R}$ defined by $f(x) := 1/\sqrt{x}$ for $0 < x \le 1$ and $f(0) := 0$ is unbounded and hence not Riemann integrable on $[0,1]$. Show that the *improper* integral of f on $[0,1]$ is $\int_0^1 f(x)\,dx = 2$.

Solution:

For any $c > 0$ we have $\int_c^1 f(x)\,dx = \int_c^1 (1/\sqrt{x})\,dx = 2\sqrt{x}\,|_c^1 = 2\,(1-\sqrt{c})$. Hence, letting $c \to 0+$, we obtain that the improper integral of f on $[0,1]$ is $\int_0^1 f(x)\,dx = \lim_{c\to 0+} 2\,(1-\sqrt{c}) = 2$.

2. With A an $n \times n$ matrix, the formula $y = Ax$ can be regarded as instructions to substitute, for each of the n variables y_i, the corresponding linear combination of the variables x_j. Suppose this is combined with another such substitution, $x = Bz$, so that ultimately each variable y_i is substituted-for by an expression in the variables z_k. What matrix represents this last substitution?

Solution:

For this very simple problem, one could proceed component-wise, but it is simpler to observe, from $y = Ax = A(Bz) = (AB)z$, that the desired matrix is just AB. [Historically, this is the reason why matrix multiplication is defined the way it is; it also exhibits how the operation of composition (for linear transformations) is in a technical sense "represented" by the operation of multiplication (for matrices).] Note that the restriction to square matrices in this problem was unnecessary.

3. Let $\lambda \in (0,\infty)$. Consider the joint probability density function $f_{X,Y}$ given by

$$f_{X,Y}(x,y) := \lambda^2 e^{-\lambda y} \text{ for } 0 \le x \le y < \infty$$

and $f_{X,Y}(x,y) := 0$ elsewhere.

(a) Show that the marginal distribution of X is exponential with (inverse-scale) parameter λ.

(b) Show that the marginal distribution of Y is gamma with index parameter 2 and inverse-scale parameter λ.

Solution:

(a) $f_X(x) = \int_x^\infty \lambda^2 e^{-\lambda y}\,dy = \lambda e^{-\lambda x}$, $x \ge 0$.

(b) $f_Y(y) = \int_0^y \lambda^2 e^{-\lambda y}\, dx = \lambda^2 y e^{-\lambda y}$, $y \geq 0$.

4. Let $n \geq 2$ be an integer and let $N := \{1, 2, \ldots, n\}$. How many different trees T can we form having $V(T) = N$ and vertex 1 as a leaf? Justify your answer.

Solution:

The answer is $(n-1)^{n-2}$. Because vertex 1 is a leaf, $T' = T - \{1\}$ is a tree and, by Cayley's formula, there are $(n-1)^{n-3}$ different trees T'. Then, for each choice of T', there are $n-1$ choices for a neighbor for vertex 1. Hence, there are $(n-1) \cdot (n-1)^{n-3} = (n-1)^{n-2}$ such trees.

5. Prove that if an $n \times n$ matrix A can be partitioned in the form

$$A = \begin{bmatrix} E & Z \\ G & H \end{bmatrix},$$

where E is a square matrix and Z consists entirely of zeros, then $\det A = (\det E)(\det H)$.

Solution:

Recall that $\det A$ is a signed sum of all products of n factors which can be formed by selecting exactly one factor from each row and each column of A. Suppose that E is $m \times m$. Then because $Z = 0$, the m factors from the first m columns of A for such a product to be nonzero must come from E; this also uses up the first m rows of A. The remaining $n - m$ factors must come appropriately from the entries of H. When signs are taken into account (using the fact that sign is a multiplicative function on the symmetric group), we obtain the desired result.

Solutions can also be based upon induction on the size of E or on the size of H.

6. Let V be the linear subspace of $\mathbb{R}^3$ spanned by the (column) vectors $v = [1, 1, 0]^T$ and $w = [0, 1, 1]^T$.

(a) Find an orthonormal basis $\{e, f\}$ for V (with respect to the usual inner product).

(b) Let $g = \left[\sqrt{3}/3, -\sqrt{3}/3, \sqrt{3}/3\right]^T$. Verify that $\{e, f, g\}$ forms an orthonormal basis of $\mathbb{R}^3$.

(c) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ denote the projection map onto V, and suppose A is the matrix for T with respect to the usual basis of $\mathbb{R}^3$. What is the rank of A?

Solution:

(a) Using the Gram–Schmidt process, we obtain an orthonormal basis $\{e, f\}$, where

$$e = \frac{1}{\sqrt{2}}[1,1,0]^T = \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right]^T$$

and

$$w - \langle w, e\rangle e = [0,1,1]^T - \frac{\sqrt{2}}{2}\left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right]^T = \left[-\frac{1}{2}, \frac{1}{2}, 1\right]^T,$$

and

$$\left\| \left[-\frac{1}{2}, \frac{1}{2}, 1\right]^T \right\| = \sqrt{\frac{3}{2}},$$

so

$$f = \left[-\frac{1}{2}, \frac{1}{2}, 1\right]^T \Big/ \left\| \left[-\frac{1}{2}, \frac{1}{2}, 1\right]^T \right\| = \left[-\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}\right]^T.$$

(b) The vector g has unit length and is proportional to $[1,-1,1]^T$ which is easily seen to be orthogonal to v and w.

(c) The matrix has rank 2 since its columns span V.

7. Consider the following linear program:

$$\begin{array}{ll} \text{Minimize} & c^T x \\ \text{subject to} & Ax \geq b, \quad x \geq 0. \end{array}$$

If it is known that the objective function of the problem is unbounded below in the feasible region, can you change the vector b to make the problem solvable? Justify your answer.

Solution:

By the duality theorem, the unboundedness of the linear program implies the infeasibility of its dual, which is

$$\begin{array}{ll} \text{Maximize} & b^T w \\ \text{subject to} & A^T w \leq c, \quad w \geq 0. \end{array}$$

The dual feasible region $\{w : A^T w \leq c,\ w \geq 0\}$ is independent of the vector b; it will remain infeasible for any vector b. By the duality theorem again, no matter how the vector b is chosen, the primal linear program is either infeasible or unbounded.

8. Let X be a Poisson random variable with parameter λ. Show that $\mathbb{P}(X = i)$ increases monotonically and then decreases monotonically as i increases, reaching its maximum value when i is the largest integer not exceeding λ.

Solution:

Consider the ratio $\mathbb{P}(X = i)/\mathbb{P}(X = i-1)$, which, after cancellation, is λ/i. The frequency function is increasing when the ratio is greater than unity, i.e., when $i < \lambda$, and decreasing when the ratio is less than unity, i.e., when $i > \lambda$. Thus, the largest value occurs when i is the integer part of λ. (NOTE: If λ is an integer, then $i = \lambda - 1$ and $i = \lambda$ both give the maximum.)

9. Let $f(x)$ be continuously differentiable, strictly increasing, and concave for $x \geq 0$, with $f(0) = 0$. Show that $d(x, y) := f(|x - y|)$ defines a metric on the nonnegative reals.

Solution:

Clearly $d(x, y) = 0$ if and only if $x = y$, and $d(x, y) = d(y, x)$. The triangle inequality follows from showing that $f(a + b) \leq f(a) + f(b)$ for any $a, b \geq 0$. From continuous differentiability,

$$\begin{aligned} f(a+b) &= \int_0^{a+b} f'(x)dx \\ &= \int_0^a f'(x)dx + \int_a^{a+b} f'(x)dx \\ &= f(a) + \int_a^{a+b} f'(x)dx \\ &\leq f(a) + \int_0^b f'(x)dx \\ &= f(a) + f(b), \end{aligned}$$

where the inequality is due to the fact that $f'(x)$ is decreasing.

10. When a current I (measured in amperes) flows through a resistance R (measured in ohms), the power generated is given by by $W = I^2R$ (measured in watts). Suppose that I and R are independent random variables with densities

$$f_I(x) = 6x(1 - x)\, \mathbb{I}\{0 \leq x \leq 1\}$$

and

$$f_R(y) = 2y\, \mathbb{I}\{0 \leq y \leq 1\}.$$

Determine the distribution function of W.

Solution #1:

For $0 < w < 1$,

$$F_W(w) := \mathbb{P}(W \le w) = \iint 6x(1-x)\, 2y\, dy\, dx,$$

where the integration is over the region $x^2 y \le w$, $0 \le x \le 1$, $0 \le y \le 1$.

Continuing (NOTE: for some students it may be useful to graph the region to help determine limits of integration),

$$\begin{aligned} F_W(w) &= \int_0^{\sqrt{w}} \int_0^1 12x(1-x)y\, dy\, dx + \int_{\sqrt{w}}^1 \int_0^{w/x^2} 12x(1-x)y\, dy\, dx \\ &= \left(3w - 2w^{3/2}\right) + \left(3w - 6w^{3/2} + 3w^2\right) \\ &= 6w - 8w^{3/2} + 3w^2. \end{aligned}$$

Solution #2:

By standard change-of-variables calculations, the joint density of R and W is

$$f_{R,W}(r,w) = 6\left(1 - \sqrt{\frac{w}{r}}\right) \mathbb{I}\{0 \le w \le r \le 1\}.$$

Integrating over r gives, for $0 \le w \le 1$,

$$f_W(w) = 6\int_w^1 \left(1 - \sqrt{\frac{w}{r}}\right) dr = 6\left[(1-w) - 2w^{1/2}\left(1 - w^{1/2}\right)\right] = 6\left(1 - 2w^{1/2} + w\right).$$

Then for $0 \le w \le 1$ we find $F_W(w) = 6w - 8w^{3/2} + 3w^2$.

11. Let $B(m,n)$ denote the complete bipartite graph with m vertices in one part and n vertices in the other. Consider the graph $G = B(4,4)$. With justification, determine

(a) how many cycles of length 5 are contained in G, and

(b) how many cycles of length 4.

Solution:

(a) Since one characterization of bipartite graphs is as graphs with no odd-length cycles, part (a) has the immediate answer "zero."

(b) Let X and Y be the two "parts" of G, so that any cycle must alternate between X and Y. The critical fact, not true for (say) 6-cycles, is that a bipartite 4-cycle C is (why?) uniquely determined by its set of vertices. By completeness, the two X-vertices of C can be chosen in any of $\binom{4}{2} = 6$ ways and, independently of that, the two Y-vertices of C can also be chosen in any of 6 ways. So the number of cycles of length 4 is 36.

12. Find all the values of α and β such that the following problem has an optimal solution with basic variables x_3 and x_4:

$$\begin{array}{lrcrcrcrcr}
\text{Minimize} & \alpha x_1 & + & 13x_2 & - & \alpha x_3 & + & 3x_4 & & \\
\text{subject to} & x_1 & + & x_2 & - & 2x_3 & + & 3x_4 & = & -1 \\
 & 2x_1 & - & x_2 & - & 3x_3 & + & 4x_4 & = & \beta \\
 & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0.
\end{array}$$

Solution:

To move the variables x_3 and x_4 to the basis, we consider their corresponding columns in the system to form the matrix $B = \begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix}$. Premultiply the linear system by $B^{-1} = \begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}$ and compute the reduced costs to get the following tableau:

	$3-\alpha$	$7\alpha-2$	0	0
$-4-3\beta$	-2	7	1	0
$-3-2\beta$	-1	5	0	1

If this is a tableau for an optimal solution then we have

$$-4-3\beta \geq 0, \quad -3-2\beta \geq 0, \quad 3-\alpha \geq 0, \quad 7\alpha - 2 \geq 0.$$

Therefore α and β should have values in the following ranges:

$$\tfrac{2}{7} \leq \alpha \leq 3 \quad \text{and} \quad \beta \leq -\tfrac{3}{2}.$$

13. Let X be a real-valued random variable with density f, and let $Y_1, \dots, Y_n$ be Bernoulli variables which are conditionally independent given X. We assume that $p(x) := \mathbb{P}(Y_k = 1 \,|\, X = x)$ is independent of k.

Compute the conditional density of X given $Y_1 = y_1, \dots, Y_n = y_n$ as a function of f, p, and n_y, where n_y is defined as the number of values k such that $y_k = 1$.

Solution:

Denote this conditional density by $x \mapsto f(x|y_1, \dots, y_n)$. It is defined by requiring the following for any interval A:

$$P(X \in A \,|\, Y_1 = y_1, \dots, Y_n = y_n) = \int_A f(x|y_1, \dots, y_n)\, dx.$$

The computation

$$\begin{aligned}\mathbb{P}(X \in A,\, Y_1 = y_1,\, \ldots,\, Y_n = y_n) &= \int_A \mathbb{P}(Y_1 = y_1 \,|\, X = x) \cdots \mathbb{P}(Y_n = y_n \,|\, X = x) f(x)\, dx \\ &= \int_A p(x)^{n_y}(1 - p(x))^{n-n_y} f(x)\, dx\end{aligned}$$

leads directly to

$$f(x|y_1, \ldots, y_n) = \frac{p(x)^{n_y}(1 - p(x))^{n-n_y} f(x)}{\int_{-\infty}^{\infty} p(u)^{n_y}(1 - p(u))^{n-n_y} f(u)\, du}.$$

14. Estimate the probability that in 60 independent tosses of a pair of fair dice the sum is never equal to 4. Use the Poisson approximation to the binomial distribution.

Solution:

Let $X_n := 1$ if the sum obtained on the nth toss is 4 and let $X_n := 0$ otherwise, $n = 1, 2, \ldots, 60$. Then $X_1, \ldots, X_{60}$ are independent Bernoulli random variables with $\mathbb{P}(X_n = 1) = 3/36 = 1/12$ for each n, and therefore $S := \sum_{n=1}^{60} X_n$ is binomial with sample size 60 and success probability $1/12$. By the Poisson approximation, the distribution of S is approximately Poisson with mean $\lambda = 60/12 = 5$. Consequently,

$$\mathbb{P}(S = 0) \approx e^{-5}.$$

(Note: The exact value is $\mathbb{P}(S = 0) = (11/12)^{60} = 0.0054\ldots$, whereas $e^{-5} = 0.0067\ldots$. Of course, students taking the exam will not have calculators.)

15. Show that a unitary $n \times n$ matrix U which is also upper triangular must be diagonal.

Solution:

Writing out the unitarity condition $U^*U = I$, with I the $n \times n$ identity, in terms of components gives the conditions

$$\begin{array}{rll} |u_{11}|^2 = 1; & \overline{u}_{11}u_{1k} = 0, & k > 1; \\ |u_{12}|^2 + |u_{22}|^2 = 1; & \overline{u}_{12}u_{1k} + \overline{u}_{22}u_{2k} = 0, & k > 2; \\ |u_{13}|^2 + |u_{23}|^2 + |u_{33}|^2 = 1; & \overline{u}_{13}u_{1k} + \overline{u}_{23}u_{2k} + \overline{u}_{33}u_{3k} = 0, & k > 3; \end{array}$$

and so forth. These equations may be solved successively to give

$$|u_{kk}| = 1, \quad k = 1, ..., n; \qquad u_{\ell,k} = 0, \quad \ell < k.$$

A solution can also be based upon induction on the size of U.

16. Let f be a nonnegative continuous function on a compact interval $J = [a, b]$ with $a < b$, and let $M := \sup\{f(x) : x \in J\}$. Prove that if M_p is defined by

$$M_p := \left\{ \int_a^b [f(x)]^p \, dx \right\}^{1/p}$$

for $0 < p < \infty$, then $M = \lim_{p\to\infty} M_p$. Do *not* make any appeal to theory of L^p-spaces.

Solution:

First note that $0 \le M < \infty$ and that M is achieved by f. Direct estimation shows that $M_p \le M(b-a)^{1/p}$, so $\limsup_{p\to\infty} M_p \le M$. Conversely, given $\epsilon > 0$ we have $f(x) \ge M - \epsilon$ on some subinterval $[c, d]$ of J with $c < d$; then

$$M_p \ge \left\{ \int_c^d [f(x)]^p \, dx \right\}^{1/p} \ge \left\{ \int_c^d (M-\epsilon)^p \, dx \right\}^{1/p} = (M-\epsilon)(d-c)^{1/p},$$

so $\liminf_{p\to\infty} M_p \ge M - \epsilon$. Letting ϵ drop to 0, we find

$$M \le \liminf_{p\to\infty} M_p \le \limsup_{p\to\infty} M_p \le M,$$

and the desired result follows.

17. Let $S = \{1, 4, 9, 16, 25, \dots\}$ be the set of squared positive integers, and define

$$x_i := \begin{cases} 0 & \text{if } i \notin S \\ 1 & \text{if } i \in S. \end{cases}$$

(a) Compute $\liminf_{i\to\infty} x_i$ and $\limsup_{i\to\infty} x_i$.

(b) Evaluate $\displaystyle\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i$.

(c) Evaluate $\displaystyle\lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} \sqrt{i}$.

Solution:

(a) The sequence (x_i) is a 0/1-valued sequence with a subsequence of 0s and a subsequence of 1s, so the lim inf is 0 and the lim sup is 1.

(b) The nonnegative sum $\sum_{i=1}^{n} x_i$, which counts the number of perfect squares less than or equal to n, clearly does not exceed $\sqrt{n}$. Thus the limit in question vanishes.

(c) We have

$$0 \le \sum_{i=1}^{n} \sqrt{i} \le \sum_{i=1}^{n} \sqrt{n} = n^{3/2},$$

so again the desired limit is 0.

18. Let n, a, b be nonnegative integers with $n \ge a+b$. Give a combinatorial proof of the following identity:

$$\binom{n}{a}\binom{n-a}{b} = \binom{n}{a+b}\binom{a+b}{a}.$$

NOTE: It is *not* acceptable to convert the binomial coefficients into expressions involving factorials and proceed by algebraic methods.

Solution #1:

Let N be an n-element set. We ask: In how many ways can we choose an ordered pair (A, B) of disjoint subsets of N with $|A| = a$ and $|B| = b$?

On the one hand, we can select the elements of A in $\binom{n}{a}$ ways. For each such choice, there are $\binom{n-a}{b}$ ways to choose the elements of B, giving $\binom{n}{a}\binom{n-a}{b}$ pairs in total.

On the other hand, we can begin by choosing the elements for $A \cup B$ in $\binom{n}{a+b}$ ways and for each such choice, select which elements of this union are in A in $\binom{a+b}{a}$ ways, giving $\binom{n}{a+b}\binom{a+b}{a}$ choices.

Since $\binom{n}{a}\binom{n-a}{b}$ and $\binom{n}{a+b}\binom{a+b}{a}$ are both correct answers to the same question, they are equal.

Solution #2:

(This second solution is essentially the same as the first, but in a different guise.)

Let N be an n-element set.

Let $\mathcal{X}$ denote the set of ordered pairs (A, B) where $|A| = a$, $|B| = b$, $A \cap B = \emptyset$, and $A, B \subseteq N$. Clearly $|\mathcal{X}| = \binom{n}{a}\binom{n-a}{b}$ (as in Solution 1).

Let $\mathcal{Y}$ denote the set of ordered pairs (S, T) where $|S| = a$, $|T| = a+b$, and $S \subseteq T \subseteq N$. Clearly $|\mathcal{Y}| = \binom{n}{a+b}\binom{a+b}{a}$ (choose T and then choose S).

Let $f : \mathcal{X} \to \mathcal{Y}$ by $f(A, B) = (A, A \cup B)$. Note:

- Since $(A,B) \in \mathcal{X}$, it follows that $|A \cup B| = a + b$ and so $f(A,B)$ is indeed an element of $\mathcal{Y}$.
- f is one-to-one because if $f(A,B) = f(A',B')$ then $A = A'$ and $A \cup B = A' \cup B'$. It follows that $B = B'$ by deleting the elements of $A = A'$ from both sides and using the fact that A, B and A', B' are both disjoint pairs.
- f is onto because if $(S,T) \in \mathcal{Y}$ then $(S, T - S) \in \mathcal{X}$ and $f(S, T - S) = (S,T)$.

Therefore f is a bijection between $\mathcal{X}$ and $\mathcal{Y}$, and so $|\mathcal{X}| = |\mathcal{Y}|$ and the result follows.

19. Suppose that f is a continuous function from $\mathbb{R}$ into $\mathbb{R}$ and that K is a compact subset of $\mathbb{R}$. Prove that f either has a zero in K or else is strictly bounded away from zero on K, i.e., satisfies $|f(x)| \geq \epsilon$ for some $\epsilon > 0$.

Solution #1:

If f is not bounded away from 0 on K, then there must exist a sequence $x_n \in K$ such that $|f(x_n)| < 1/n$ for each positive integer n. Because K is compact, there is a convergent subsequence $x_{n_k} \to x_* \in K$. Since $|f(x_{n_k})| < 1/n_k$ and f is continuous, taking $k \to \infty$ gives $f(x_*) = 0$.

Solution #2:

The continuous image $f(K)$ is also compact, hence closed, and its complement $f(K)^c$ is open. Thus, if $0 \in f(K)^c$, then there exists some open ball $B(0,\epsilon) = \{x : |x| < \epsilon\}$ with $\epsilon > 0$ such that $B(0,\epsilon) \subset f(K)^c$. Thus, either $0 \in f(K)$ or else $f(K) \subset B(0,\epsilon)^c$.

Solution #3:

The continuous function $|f|$ assumes a minimum value on K. Since $|f|$ is nonnegative, this minimum value is either 0 or else some $\epsilon > 0$. These two possibilities yield, respectively, the alternative situations stated in the problem.

20. Consider the following matrix (with $n \geq 1$):

$$A = \begin{bmatrix} 0 & 1 & 2 & \dots & n \\ 1 & 2 & 3 & \dots & n+1 \\ 2 & 3 & 4 & \dots & n+2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n+1 & n+2 & \dots & 2n \end{bmatrix}.$$

(a) Find two vectors u and v such that for any x there exist $a, b \in \mathbb{R}$ such that $Ax = au + bv$.

(b) Compute the eigenvalues of A, together with their (algebraic) multiplicities.

[HINT: A useful formula for solving Problem 20 is $1+4+\cdots+n^2 = n(n+1)(2n+1)/6$. You may use this formula without proving it.]

Solution:

(a) Let $u := (1, 1, \ldots, 1)^T$ and $v := (0, 1, 2, \ldots, n)^T$. The jth column in A is $a_j = (j-1)u + v$. This implies that, for any x,

$$Ax = \sum_{j=1}^{n+1} x_j a_j = \left[\sum_{j=1}^{n+1} (j-1)x_j\right] u + \left(\sum_{j=1}^{n+1} x_j\right) v.$$

(b) The matrix A is symmetric and thus diagonalizable with orthogonal eigenspaces. The eigenspace for the eigenvalue 0 is characterized by $\sum_{j=1}^{n+1}(j-1)x_j = \sum_{j=1}^{n+1} x_j = 0$ and so is a space of codimension 2 (i.e., dimension $n-1$), the vectors u and v being shown linearly independent by computing the first 2-by-2 determinant. To compute the last two eigenvalues, it suffices to compute the matrix of A with respect to u and v, i.e., to express $Au = au + cv$, $Av = bu + dv$. Using the formula in the hint,

$$\begin{aligned} a &= 0 + 1 + \cdots + n = n(n+1)/2 \\ c &= n+1 \\ b &= 0^2 + 1^2 + \cdots + n^2 = n(n+1)(2n+1)/6 \\ d &= n(n+1)/2 \end{aligned}$$

The problem thus reduces to solving for the eigenvalues of

$$M := \frac{n+1}{6} \begin{bmatrix} 3n & 6 \\ n(2n+1) & 3n \end{bmatrix}.$$

Those are obtained by finding the roots of

$$t^2 - n(n+1)t - \frac{n(n+1)^2(n+2)}{12}$$

and are thus

$$\lambda_1 = (n+1)\left[\frac{n}{2} + \sqrt{\frac{n(2n+1)}{6}}\right], \qquad \lambda_2 = (n+1)\left[\frac{n}{2} - \sqrt{\frac{n(2n+1)}{6}}\right].$$