Department of Applied Mathematics and Statistics
The Johns Hopkins University

INTRODUCTORY EXAMINATION

Friday, January 28, 2005

Instructions: Read carefully!

1. This **closed-book** examination consists of 20 problems (sorry, no choices), each worth 5 points. The passing grade has been set at $66\frac{2}{3}\%$. Partial credit will be given as appropriate; each part of a problem will be given the same weight unless otherwise specified. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. However, our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been grouped by topic, but there are roughly equally many mainly motivated by each of the four areas identified in the syllabus (linear algebra; real analysis; probability; discrete mathematics and operations research/optimization). Nor have the problems been arranged systematically by difficulty, although you may find the first three easier than most of the others. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM-NUMBER on each sheet.

5. The examination will begin at 8:30 AM; lunch and refreshments will be provided. The exam will end just before 5:00 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

GOOD LUCK!

1. The function $f : [0,1] \to \mathbb{R}$ defined by $f(x) := 1/\sqrt{x}$ for $0 < x \leq 1$ and $f(0) := 0$ is unbounded and hence not Riemann integrable on $[0,1]$. Show that the *improper* integral of f on $[0,1]$ is $\int_0^1 f(x)\,dx = 2$.

2. With A an $n \times n$ matrix, the formula $y = Ax$ can be regarded as instructions to substitute, for each of the n variables y_i, the corresponding linear combination of the variables x_j. Suppose this is combined with another such substitution, $x = Bz$, so that ultimately each variable y_i is substituted-for by an expression in the variables z_k. What matrix represents this last substitution?

3. Let $\lambda \in (0, \infty)$. Consider the joint probability density function $f_{X,Y}$ given by

$$f_{X,Y}(x,y) := \lambda^2 e^{-\lambda y} \text{ for } 0 \leq x \leq y < \infty$$

and $f_{X,Y}(x,y) := 0$ elsewhere.

(a) Show that the marginal distribution of X is exponential with (inverse-scale) parameter λ.

(b) Show that the marginal distribution of Y is gamma with index parameter 2 and inverse-scale parameter λ.

4. Let $n \geq 2$ be an integer and let $N := \{1, 2, \ldots, n\}$. How many different trees T can we form having $V(T) = N$ and vertex 1 as a leaf? Justify your answer.

5. Prove that if an $n \times n$ matrix A can be partitioned in the form

$$A = \begin{bmatrix} E & Z \\ G & H \end{bmatrix},$$

where E is a square matrix and Z consists entirely of zeros, then $\det A = (\det E)(\det H)$.

6. Let V be the linear subspace of $\mathbb{R}^3$ spanned by the (column) vectors $v = [1,1,0]^T$ and $w = [0,1,1]^T$.

(a) Find an orthonormal basis $\{e, f\}$ for V (with respect to the usual inner product).

(b) Let $g = \left[\sqrt{3}/3, -\sqrt{3}/3, \sqrt{3}/3\right]^T$. Verify that $\{e, f, g\}$ forms an orthonormal basis of $\mathbb{R}^3$.

(c) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ denote the projection map onto V, and suppose A is the matrix for T with respect to the usual basis of $\mathbb{R}^3$. What is the rank of A?

7. Consider the following linear program:

$$\begin{array}{ll} \text{Minimize} & c^T x \\ \text{subject to} & Ax \geq b, \quad x \geq 0. \end{array}$$

If it is known that the objective function of the problem is unbounded below in the feasible region, can you change the vector b to make the problem solvable? Justify your answer.

8. Let X be a Poisson random variable with parameter λ. Show that $\mathbb{P}(X = i)$ increases monotonically and then decreases monotonically as i increases, reaching its maximum value when i is the largest integer not exceeding λ.

9. Let $f(x)$ be continuously differentiable, strictly increasing, and concave for $x \geq 0$, with $f(0) = 0$. Show that $d(x, y) := f(|x - y|)$ defines a metric on the nonnegative reals.

10. When a current I (measured in amperes) flows through a resistance R (measured in ohms), the power generated is given by by $W = I^2 R$ (measured in watts). Suppose that I and R are independent random variables with densities

$$f_I(x) = 6x(1-x)\,\mathbb{I}\{0 \leq x \leq 1\}$$

and

$$f_R(y) = 2y\,\mathbb{I}\{0 \leq y \leq 1\}.$$

Determine the distribution function of W.

11. Let $B(m,n)$ denote the complete bipartite graph with m vertices in one part and n vertices in the other. Consider the graph $G = B(4,4)$. With justification, determine

(a) how many cycles of length 5 are contained in G, and

(b) how many cycles of length 4.

12. Find all the values of α and β such that the following problem has an optimal solution with basic variables x_3 and x_4:

$$\begin{array}{lrcrcrcrcr} \text{Minimize} & \alpha x_1 & + & 13x_2 & - & \alpha x_3 & + & 3x_4 & & \\ \text{subject to} & x_1 & + & x_2 & - & 2x_3 & + & 3x_4 & = & -1 \\ & 2x_1 & - & x_2 & - & 3x_3 & + & 4x_4 & = & \beta \\ & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0. \end{array}$$

13. Let X be a real-valued random variable with density f, and let $Y_1, \ldots, Y_n$ be Bernoulli variables which are conditionally independent given X. We assume that $p(x) := \mathbb{P}(Y_k = 1 \mid X = x)$ is independent of k.

Compute the conditional density of X given $Y_1 = y_1, \ldots, Y_n = y_n$ as a function of f, p, and n_y, where n_y is defined as the number of values k such that $y_k = 1$.

14. Estimate the probability that in 60 independent tosses of a pair of fair dice the sum is never equal to 4. Use the Poisson approximation to the binomial distribution.

15. Show that a unitary $n \times n$ matrix U which is also upper triangular must be diagonal.

16. Let f be a nonnegative continuous function on a compact interval $J = [a,b]$ with $a < b$, and let $M := \sup\{f(x) : x \in J\}$. Prove that if M_p is defined by

$$M_p := \left\{ \int_a^b [f(x)]^p \, dx \right\}^{1/p}$$

for $0 < p < \infty$, then $M = \lim_{p \to \infty} M_p$. Do *not* make any appeal to theory of L^p-spaces.

17. Let $S = \{1, 4, 9, 16, 25, \dots\}$ be the set of squared positive integers, and define

$$x_i := \begin{cases} 0 & \text{if } i \notin S \\ 1 & \text{if } i \in S. \end{cases}$$

(a) Compute $\liminf_{i\to\infty} x_i$ and $\limsup_{i\to\infty} x_i$.

(b) Evaluate $\displaystyle\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i$.

(c) Evaluate $\displaystyle\lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} \sqrt{i}$.

18. Let n, a, b be nonnegative integers with $n \geq a + b$. Give a combinatorial proof of the following identity:

$$\binom{n}{a}\binom{n-a}{b} = \binom{n}{a+b}\binom{a+b}{a}.$$

NOTE: It is *not* acceptable to convert the binomial coefficients into expressions involving factorials and proceed by algebraic methods.

19. Suppose that f is a continuous function from $\mathbb{R}$ into $\mathbb{R}$ and that K is a compact subset of $\mathbb{R}$. Prove that f either has a zero in K or else is strictly bounded away from zero on K, i.e., satisfies $|f(x)| \geq \epsilon$ for some $\epsilon > 0$.

20. Consider the following matrix (with $n \geq 1$):

$$A = \begin{bmatrix} 0 & 1 & 2 & \dots & n \\ 1 & 2 & 3 & \dots & n+1 \\ 2 & 3 & 4 & \dots & n+2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n+1 & n+2 & \dots & 2n \end{bmatrix}.$$

(a) Find two vectors u and v such that for any x there exist $a, b \in \mathbb{R}$ such that $Ax = au + bv$.

(b) Compute the eigenvalues of A, together with their (algebraic) multiplicities.

[HINT: A useful formula for solving Problem 20 is $1+4+\cdots+n^2 = n(n+1)(2n+1)/6$. You may use this formula without proving it.]