Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Fall Session

Wednesday, August 24, 2011

Instructions: Read carefully!

1. This **closed-book** examination consists of 15 problems, each worth 5 points. The passing grade has been set at 50 points, i.e., 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been grouped by topic, but there are roughly equally many mainly motivated by each of the three areas identified in the syllabus (linear algebra; real analysis; probability). Nor have the problems been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. The examination will begin at 8:30 AM; lunch and refreshments will be provided. The exam will end just before 5:00 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. For any subset $A \subseteq \mathbb{N}$, the set of natural numbers, one defines its *upper asymptotic density* by

$$\bar{d}(A) = \limsup_{n\to\infty} \frac{1}{n}\text{card}(A \cap \{1, .., n\}).$$

Is $\bar{d}$ a probability measure on the set of all subsets of $\mathbb{N}$? Explain why or why not.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous one-to-one function. Is it true that f must be either strictly increasing or strictly decreasing? If so, prove this; if not, give an explicit counterexample.

3. Let $x_1, \cdots, x_n$ be linearly independent column vectors in $\mathbb{R}^n$ and A be a $n \times n$ nonsingular real matrix. Suppose $y_1, \cdots, y_n$ are vectors satisfying:

$$y_i^T A x_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Show (a) $y_1, \cdots, y_n$ are linearly independent; (b) $A^{-1} = x_1 y_1^T + \cdots + x_n y_n^T$.

4. Professor Priebe is searching frantically for his lecture notes, which might be in one of n possible boxes. For $j = 1, ..., n$, let L_j be the event that he fails to discover his lecture notes after desperately (and incompletely) rummaging through box j, and let B_j be the event that the lecture notes are in fact in box j. Suppose that $P(B_j) > 0$ for each j, and that $P(L_j|B_j) < 1$. Show that

$$P(B_j|L_j) < P(B_j) < P(B_j|L_i), \quad i \neq j.$$

5. Let $a_0, a_1, \cdots, a_n$ be complex numbers. Consider the square Vandermonde matrix

$$A = \begin{bmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^n \\ 1 & a_1 & a_1^2 & \cdots & a_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^n \end{bmatrix}.$$

(i) Show that $\det(A) = \prod_{n\geq i>j\geq 0}(a_i - a_j)$

(ii) If $a_0, a_1, a_2, \cdots, a_n$ are all distinct, and $b_0, b_1, \cdots, b_n$ are complex numbers, use part (i) to show that there is a unique polynomial f of degree n with complex coefficients such that $f(a_0) = b_0, f(a_1) = b_1, \ldots, f(a_n) = b_n$.

6. Prove that $(n/e)^n < n!$ for $n = 1, 2, 3, \ldots$.

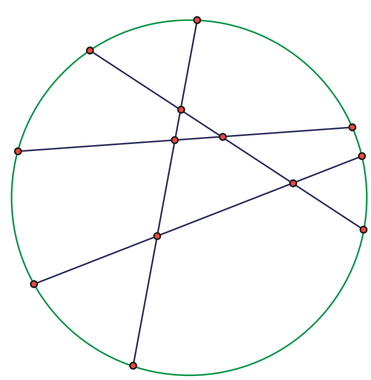
7. Let X_1 and X_2 be two iid random variables with the uniform distribution over $[0, 1]$. Define $Y = \min(X_1, X_2)$ and $Z = \max(X_1, X_2)$. Compute the conditional probability $P\{Y \geq y | Z \leq z\}$, and the conditional expectation $\mathbb{E}\{Y | Z > z\}$.

8. Let $((x_1, y_1), \cdots, (x_K, y_K))$ and $((w_1, z_1), \cdots, (w_K, z_K))$ be two sets of K points in $\mathbb{R}^2$. We will fix the first and rotate the second around the origin to minimize the root sum of squared distances between the points.

   (a) Let (u_k, v_k) be the rotation of (w_k, z_k) by angle θ (counterclockwise). Note that u_k and v_k are both functions of w_k, z_k, θ; that is, $u_k = u_k(w_k, z_k, \theta)$ and $v_k = v_k(w_k, z_k, \theta)$. Write (u_k, v_k) in terms of w_k, z_k, and θ.

   (b) Find θ^* to minimize $d_\theta = \sqrt{(u_1 - x_1)^2 + (v_1 - y_1)^2 + \cdots + (u_K - x_K)^2 + (v_K - y_K)^2}$ in terms of the x_k, y_k, w_k, z_k.

9. Let n be a positive integer and let $A_1, B_1, A_2, B_2, \ldots, A_n, B_n$ be $2n$ points chosen independently and uniformly on the circumference of a circle. Draw the n line segments $\overline{A_i B_i}$ (for $1 \leq i \leq n$) and let X be the number of points of intersection of these line segments. The following diagram is an example in which $n = 4$ and $X = 5$.



Evaluate (with justification) the expectation $\mathbb{E}(X)$ as a function of n.

10. If A is a bounded subset of $\mathbb{R}$ with a unique accumulation point, $a \in \mathbb{R}$, prove there is a sequence $(x_1, \ldots, x_n, \ldots)$ which converges to a such that $A = \{x_1, \ldots, x_n, \ldots\}$. (Hint: You may use the fact that a countable union of finite sets is countable.)

11. Let A be an $n \times n$ matrix that is (i) lower triangular, with (ii) entries that are all real and (iii) diagonal entries that are all strictly positive. Prove that A is invertible and that A^{-1} also has properties (i)–(iii)

12. A real-valued function f on $[0, 1]$ is said to be Hölder continuous of order α if there is a constant C such that $|f(x) - f(y)| \le C|x - y|^\alpha$. Define

$$\|f\|_\alpha = \max_x |f(x)| + \sup_{x,y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

Show that for $0 < \alpha \le 1$ the set of functions $S_\alpha = \{f \in C[0, 1] : \|f\|_\alpha \le 1\}$ is a compact subset of $C[0, 1]$ in the topology of uniform convergence.

13. Let $V \subseteq \mathbb{R}^n$ be a linear subspace of dimension k. Find the dimension of the set of linear mappings $L : \mathbb{R}^n \to \mathbb{R}^m$ whose nullspace contains V.

14. A sequence of random variables $\{X_n,\, n = 1, 2, 3 \ldots\}$ converges in distribution to a random variable X if $\lim_{n\to\infty} F_{X_n}(t) = F_X(t)$ whenever t is a point at which F_X is continuous. Give an example of a sequence $\{X_n\}$ which converges in distribution, but for which the pointwise limit $F(t) = \lim_{n\to\infty} F_{X_n}(t)$ is not a distribution function.

15. Let A be a real square matrix. Prove that $\operatorname{rank}(A^2) \le \operatorname{rank}(A)$ and give an example where the inequality is strict.