Department of Applied Mathematics and Statistics
The Johns Hopkins University

INTRODUCTORY EXAMINATION–FALL SESSION

Tuesday, August 24, 2010

Instructions: Read carefully!

1. This **closed-book** examination consists of 15 problems, each worth 5 points. The passing grade has been set at 50 points, i.e., 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been grouped by topic, but there are roughly equally many mainly motivated by each of the three areas identified in the syllabus (linear algebra; real analysis; probability). Nor have the problems been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. The examination will begin at 8:30 AM; lunch and refreshments will be provided. The exam will end just before 5:00 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Given any real-valued constant $c > 0$, find
$$\lim_{n\to\infty} \frac{1^c + 2^c + \cdots + n^c}{n^{c+1}}.$$

Solution: We have
$$\lim_{n\to\infty} \frac{1^c + 2^c + \cdots + n^c}{n^{c+1}} = \lim_{n\to\infty} \left[\frac{1}{n}\left(\frac{1}{n}\right)^c + \cdots + \frac{1}{n}\left(\frac{n}{n}\right)^c \right] = \int_0^1 x^c\, dx = \frac{1}{c+1}.$$

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2. Let D be a nonsingular diagonal matrix and $D = (I + A)^{-1}A$. Show that the matrix A is also nonsingular diagonal.

Solution: It follows that $(I + A)D = A$. Hence we have $D = A(I - D)$. Because D is nonsingular, both A and $(I - D)$ are nonsingular. Therefore, we have $A = D(I - D)^{-1}$. As a product of two nonsingular diagonal matrices, A must also be nonsingular diagonal.

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3. Let X be a random variable having density function f given by
$$f(x) = c|x|e^{-x^2/2}, \quad -\infty < x < \infty$$
for some constant c.

(a) What is the value of c?

(b) For $k = 1, 2, \ldots$, calculate EX^k explicitly.

Solution:

(a) For f to be a density function it is necessary and sufficient that
$$1 = \int_{-\infty}^{\infty} f(x)\, dx = 2c \int_0^{\infty} xe^{-x^2/2}\, dx = 2c,$$
i.e., $c = 1/2$.

(b) Since f is symmetric, the odd moments of X (which are clearly finite) vanish. Moreover, for $r = 1, 2, \ldots$ we have, using the change of variables $y = x^2/2$,

$$EX^{2r} = \int_{-\infty}^{\infty} x^{2r} f(x)\, dx = \int_0^{\infty} x^{2r+1} e^{-x^2/2}\, dx = \int_0^{\infty} (2y)^r e^{-y}\, dy = 2^r r!.$$

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4. Let A be a symmetric and positive definite $n \times n$ real matrix. Show that for any eigenvalue λ of A we have

$$\lambda > \det(A) \left(\frac{n-1}{\operatorname{trace}(A)} \right)^{n-1}.$$

HINT: Use the arithmetic-geometric mean inequality.

Solution: Let $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of A. Then applying the arithmetic-geometric mean inequality, we have

$$\frac{\det(A)}{\lambda_1} \leq \left(\frac{\lambda_2 + \cdots + \lambda_n}{n-1} \right)^{n-1} < \left(\frac{\operatorname{trace}(A)}{n-1} \right)^{n-1}.$$

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5. Given two sequences of ordered positive real numbers $w_1 \leq w_2 \leq \cdots \leq w_n$ and $x_1 \leq x_2 \leq \cdots \leq x_n$, prove that

$$\frac{\sum_{i=1}^n w_i x_i}{\sum_{i=1}^n w_i} \geq \frac{1}{n} \sum_{i=1}^n x_i.$$

Solution: Expand and rearrange the inequality

$$\sum_{i,j} (w_i - w_j)(x_i - x_j) \geq 0.$$

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6. Consider the sequence $u_k \geq 0$, $k = 1, 2, ...$, and suppose

$$\sum_{k=1}^{\infty} u_k < \infty.$$

Show that the limit

$$\lim_{n\to\infty} \prod_{k=1}^{n} (1 + u_k)$$

exists and is finite.

Solution: Since $\prod_{k=1}^{n}(1 + u_k), n = 1, 2, ...,$ is a nondecreasing sequence, the limit exists, possibly infinite. Using the inequality $1 + x \leq e^x$, valid for $x \geq 0$, we find

$$\prod_{k=1}^{n}(1 + u_k) \leq \prod_{k=1}^{n} e^{u_k} = \exp \sum_{k=1}^{n} u_k \leq \exp \sum_{k=1}^{\infty} u_k < \infty.$$

Hence the limit is finite.

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7. Let (x_n) be a sequence of positive numbers, and denote the average of the first n entries by

$$\bar{x}_n = (x_1 + \cdots + x_n)/n.$$

Let $N = (n_k)$ be a subsequence of the positive integers with $\lim_{k\to\infty}(n_{k+1}/n_k) = r > 0$. If the sequence $(\bar{x}_n)$ converges along N to x, prove that

$$\frac{x}{r} \leq \liminf \bar{x}_n \leq \limsup \bar{x}_n \leq r\,x\ .$$

Solution: For $n_k \leq n < n_{k+1}$, the positivity of the x_n yields

$$\frac{n_k}{n_{k+1}} \bar{x}_{n_k} < \bar{x}_n < \bar{x}_{n_{k+1}} \cdot \frac{n_{k+1}}{n_k}\ .$$

As $n \to \infty$, the integer k for which $n_k \leq n < n_{k+1}$ tends to $+\infty$, and our assumptions imply that the left-most member converges to x/r and the right-most to rx.

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8. Suppose that if you are s minutes early for an appointment then you incur the cost cs, and if you are s minutes late then you incur the cost ks, where c and k are finite positive constants. Suppose that the travel time from where you presently are to the location of your appointment is a random variable having continuous probability density function f, distribution function F, and finite expectation. Determine the time at which you should depart if you want to minimize your expected cost.

Solution: This is an Example in Chapter 5 of Ross. Let X denote the travel time. Clearly you should depart no later than your appointment time. If you leave t minutes before your appointment, then you cost, call it C_t, is given by

$$C_t = \begin{cases} c\,(t-X) & \text{if } X \le t \\ k\,(X-t) & \text{if } X \ge t. \end{cases}$$

Therefore,

$$\begin{aligned} EC_t &= \int_0^t c\,(t-x) f(x)\,dx + \int_t^\infty k\,(x-t) f(x)\,dx \\ &= c\,tF(t) - c\int_0^t x f(x)\,dx + k\int_t^\infty x f(x)\,dx - k\,t[1-F(t)] \\ &= (k+c)\,t\,F(t) - c\int_0^t x f(x)\,dx + k\int_t^\infty x f(x)\,dx - k\,t. \end{aligned}$$

The value of t that minimizes EC_t can now be obtained by calculus. Differentiation yields

$$\frac{d}{dt}EC_t = (k+c)\,t\,f(t) + (k+c)\,F(t) - c\,t\,f(t) - k\,t\,f(t) - k = (k+c)\,F(t) - k$$

and

$$\frac{d^2}{dt^2}EC_t = (k+c)\,f(t) \ge 0,$$

so the minimizing value of t is the $[k/(k+c)]$-quantile of F, i.e., the unique value t^* of t satisfying $F(t^*) = k/(k+c)$.

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9. For each $n \ge 1$, let the function $g_n : [0,\infty) \to [0,\infty)$ be defined by

$$g_n(x) = \begin{cases} nx, & 0 \le x \le \frac{1}{n} \\ \frac{1}{nx}, & \frac{1}{n} < x < \infty. \end{cases}$$

(a) Show that $\lim_n g_n(x) = 0$ for all $x \ge 0$.

(b) Is the convergence uniform in x? Prove your answer.

Solution:

(a) Since $g_n(0) = 0$ for all n, the result is clear for $x = 0$. Given $x > 0$, choose integer $n_0 \equiv n_0(x)$ so that $x > 1/n_0$. (The smallest such n_0 is $\lfloor 1/x \rfloor + 1$.) Then $g_n(x) = 1/(nx)$ for all $n \geq n_0$, and the result follows.

(b) No, because $\sup_x |g_n(x) - 0| = \max_x g_n(x) = g_n(1/n) = 1$ does not vanish in the limit.

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10. A parallel system functions if and only if at least one of its components is functioning. A device consists of n components in parallel. The lifetimes $L_1, \ldots, L_n$ of the components are independent and exponentially distributed with the same rate λ. Let L be the lifetime of the system.

(a) Compute

$$P\{L \leq t\}, \quad t \geq 0.$$

(b) Show that L can be represented as

$$L = W_1 + \cdots + W_n,$$

where $W_1, \ldots, W_n$ are independent exponential variables with respective parameters $n\lambda, (n-1)\lambda, \ldots, \lambda$.

Solution:

(a) Notice that when the n components are placed in parallel the lifetime of the entire system is the maximum of the lifetimes of the components, i.e.,

$$L = L_1 \vee L_2 \vee \cdots \vee L_n.$$

Then, since the maximum of a set of random variables if less than or equal to t if and only if every random variable is less than or equal to t, we write

$$\begin{aligned} P\{L \leq t\} &= P\{L_1 \leq t,\ L_2 \leq t,\ \ldots,\ L_n \leq t\} \\ &\overset{\text{ind}}{=} P\{L_1 \leq t\} P\{L_2 \leq t\} \cdots P\{L_n \leq t\} \\ &= (1 - e^{-\lambda t})^n, \quad t \geq 0. \end{aligned}$$

(b) Initially we have n components each with an exponential lifetime (with rate λ). Let W_1 be the shortest of these lifetimes; it is exponential with rate $n\lambda$.

At W_1 we have $n-1$ components alive, and their remaining lifetimes are exponentials with rate λ (memorylessness), independent of W_1 and of each other. Therefore, the waiting time W_2 to the second failure is the minimum of the remaining $n-1$ lifetimes, and thus it is exponential with parameter $(n-1)\lambda$. Continuing in this manner we have the inter-failure times $W_1, W_2, \ldots, W_{n-1}$, and W_n is the remaining lifetime of the last surviving component. Thus L is decomposed into a sum of independent exponential random variables, namely,

$$L = W_1 + W_2 + \cdots + W_n,$$

where $W_i \sim \text{Exp}((n-i+1)\lambda)$, $i = 1, 2, \ldots, n$.

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11. Let $A_1, A_2, \ldots, A_n$ be arbitrary events, and define C_k to be the event that at least k of these events occur. Show that

$$\sum_{k=1}^{n} P(C_k) = \sum_{k=1}^{n} P(A_k).$$

Hint: Define the random variable X as the number of $A_1, \ldots, A_n$ that occur. How are the two quantities related to X?

Solution: The expressions are non-random, so the student might guess that they are both $E[X]$ and then verify this guess.

Write X as a sum of indicator random variables:

$$X = \sum_{k=1}^{n} I_{A_k}.$$

Then, by linearity of expectation,

$$E[X] = \sum_{k=1}^{n} E[I_{A_k}] = \sum_{k=1}^{n} P[A_k].$$

Also, using the representation of expectation as the sum of tail probabilities,

$$E[X] = \sum_{k=1}^{n} P[X \geq k] = \sum_{k=1}^{n} P[C_k].$$

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12. Let A be an $n \times n$ nonsingular matrix, and let B denote its classical adjoint. That is, $B = C^T$, where $C = [c_{ij}]$ is the matrix of cofactors:

$$c_{ij} = (-1)^{i+j} \det A_{ij},$$

where A_{ij} is the submatrix of A obtained by deleting the ith row and jth column. Prove that $\det B = (\det A)^{n-1}$.

Solution: The key is to recall that $A^{-1} = (\det A)^{-1} B$, and then the proof is easy. Indeed, by the multiplicativity (at the first equality below) and multilinearity (at the second equality) of the determinant we then have

$$(\det A)^{-1} = \det A^{-1} = (\det A)^{-n} \det B,$$

which rearranges to the desired result.

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13. Let $B = \left[\begin{smallmatrix} 1 & x \\ x & 3 \end{smallmatrix}\right]$ where x represents a real number. For what values of x does there exist a real symmetric matrix A such that $A^2 = B$?

Solution: If the eigenvalues of A are λ_1 and λ_2, then the eigenvalues of B must be their squares and hence nonnegative. Conversely, if B has nonnegative eigenvalues, then it follows that B has a real symmetric square root:

$$B = \Gamma \Lambda \Gamma^T \text{ implies that one can take } A = \Gamma \sqrt{\Lambda} \Gamma^T.$$

Let μ_1, μ_2 be the eigenvalues of B. By considering trace and determinant we have $\mu_1 + \mu_2 = 4$ and $\mu_1 \mu_2 = 3 - x^2$. To achieve $\mu_1, \mu_2 \geq 0$ it is necessary and sufficient that $3 - x^2 \geq 0$, i.e., $|x| \leq \sqrt{3}$.

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14. Let $p(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$ be a polynomial with real coefficients, and let M denote its companion matrix

$$M = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -c_0 & -c_1 & -c_2 & \cdots & -c_{n-2} & -c_{n-1} \end{bmatrix}.$$

Let $\lambda_1, \dots, \lambda_n$ denote the roots of p counted with multiplicity, and define the Vandermonde matrix V to be

$$V := \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_{n-1} & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \cdots & \lambda_{n-1}^2 & \lambda_n^2 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \lambda_3^{n-1} & \cdots & \lambda_{n-1}^{n-1} & \lambda_n^{n-1} \end{bmatrix}.$$

Show that $MV = VD$, where D is the diagonal matrix whose diagonal entries are (in order) $\lambda_1, \dots, \lambda_n$.

Solution: Let

$$v^{(i)} = \begin{bmatrix} 1 \\ \lambda_i \\ \lambda_i^2 \\ \vdots \\ \lambda_i^{n-1} \end{bmatrix}.$$

denote the i-th column of V. Direct calculation yields

$$\begin{aligned} Mv^{(i)} &= \begin{bmatrix} \lambda_i \\ \lambda_i^2 \\ \lambda_i^3 \\ \vdots \\ \lambda_i^{n-1} \\ -c_0 - c_1\lambda_i - c_2\lambda_i^2 - \cdots - c_{n-1}\lambda_i^{n-1} \end{bmatrix} = \begin{bmatrix} \lambda_i \\ \lambda_i^2 \\ \lambda_i^3 \\ \vdots \\ \lambda_i^{n-1} \\ \lambda_i^n - p(\lambda_i) \end{bmatrix} = \begin{bmatrix} \lambda_i \\ \lambda_i^2 \\ \lambda_i^3 \\ \vdots \\ \lambda_i^{n-1} \\ \lambda_i^n \end{bmatrix} \\ &= \lambda_i v^{(i)}, \end{aligned}$$

and the result follows easily.

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15. Suppose that $X_1, X_2, X_3, \dots$ is a sequence of nonnegative random variables, and define $M_n := \max_{k \le n} X_k$.

    Throughout the problem (and your solution), we agree to use the following shorthand for expectation, where Y is a random variable and A is an event (and I_A is the indicator of A):

    $$E(Y; A) := E(Y I_A).$$

    We also conveniently drop $\{\,\}$ as the following example illustrates:

    $$E(Y; Y > y) := E(Y; \{Y > y\}) = E(Y I_{\{Y>y\}}).$$

(a) Show for all $\alpha > 0$ that

$$E(M_n; M_n > \alpha) \leq \sum_{k=1}^{n} E(X_k; X_k > \alpha).$$

(b) If $X_1, X_2, X_3, \ldots$ are supposed further to be identically distributed (but not necessarily independent) with finite expectation, use part (a) to show that

$$\lim_{n\to\infty} \tfrac{1}{n} E(M_n) = 0.$$

Solution:

(a) Since $\Omega = \bigcup_{k=1}^{n} \{M_n = X_k\}$, we can write

$$\begin{aligned} E(M_n; M_n > \alpha) &= E\left(M_n; \{M_n > \alpha\} \cup \bigcup_{k=1}^{n} \{M_n = X_k\} \right) \\ &\leq \sum_{k=1}^{n} E(M_n; M_n > \alpha,\ M_n = X_k) \\ &= \sum_{k=1}^{n} E(X_k; X_k > \alpha,\ M_n = X_k) \\ &\leq \sum_{k=1}^{n} E(X_k; X_k > \alpha), \end{aligned}$$

using finite subadditivity of indicators and monotonicity of expectation at the first inequality.

(b) For any $\alpha > 0$ we have

$$\begin{aligned} E(M_n) &= E(M_n; M_n \leq \alpha) + E(M_n; M_n > \alpha) \\ &\leq \alpha + nE(X_1; X_1 > \alpha) \end{aligned}$$

by part (a) and the identical distribution of the X_k's. Hence,

$$\limsup_{n\to\infty} \tfrac{1}{n} E(M_n) \leq E(X_1; X_1 > \alpha)$$

for all $\alpha > 0$. However, $E(X_1; X_1 > \alpha) \to 0$ as $\alpha \to \infty$, by integrability of X_1. Thus,

$$\lim_{n\to\infty} \tfrac{1}{n} E(M_n) = 0.$$
