

Department of Applied Mathematics and Statistics
The Johns Hopkins University

INTRODUCTORY EXAMINATION—FALL SESSION

Tuesday, August 24, 2010

Instructions: Read carefully!

1. This **closed-book** examination consists of 15 problems, each worth 5 points. The passing grade has been set at 50 points, i.e., $2/3$ of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.
3. The problems have not been grouped by topic, but there are roughly equally many mainly motivated by each of the three areas identified in the syllabus (linear algebra; real analysis; probability). Nor have the problems been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
4. Start your answer to each problem on a **NEW** sheet of paper. Write only on **ONE SIDE** of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your **NAME** and the **PROBLEM NUMBER** on each sheet.
5. The examination will begin at 8:30 AM; lunch and refreshments will be provided. The exam will end just before 5:00 PM. You may leave before then, but in that case you may not return.
6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
7. **No calculators of any sort are needed or permitted.**

1. Given any real-valued constant $c > 0$, find

$$\lim_{n \rightarrow \infty} \frac{1^c + 2^c + \cdots + n^c}{n^{c+1}}.$$

2. Let D be a nonsingular diagonal matrix and $D = (I + A)^{-1}A$. Show that the matrix A is also nonsingular diagonal.

3. Let X be a random variable having density function f given by

$$f(x) = c|x|e^{-x^2/2}, \quad -\infty < x < \infty$$

for some constant c .

- (a) What is the value of c ?
(b) For $k = 1, 2, \dots$, calculate EX^k explicitly.

4. Let A be a symmetric and positive definite $n \times n$ real matrix. Show that for any eigenvalue λ of A we have

$$\lambda > \det(A) \left(\frac{n-1}{\text{trace}(A)} \right)^{n-1}.$$

HINT: Use the arithmetic-geometric mean inequality.

5. Given two sequences of ordered positive real numbers $w_1 \leq w_2 \leq \cdots \leq w_n$ and $x_1 \leq x_2 \leq \cdots \leq x_n$, prove that

$$\frac{\sum_{i=1}^n w_i x_i}{\sum_{i=1}^n w_i} \geq \frac{1}{n} \sum_{i=1}^n x_i.$$

6. Consider the sequence $u_k \geq 0$, $k = 1, 2, \dots$, and suppose

$$\sum_{k=1}^{\infty} u_k < \infty.$$

Show that the limit

$$\lim_{n \rightarrow \infty} \prod_{k=1}^n (1 + u_k)$$

exists and is finite.

7. Let (x_n) be a sequence of positive numbers, and denote the average of the first n entries by

$$\bar{x}_n = (x_1 + \dots + x_n)/n.$$

Let $N = (n_k)$ be a subsequence of the positive integers with $\lim_{k \rightarrow \infty} (n_{k+1}/n_k) = r > 0$. If the sequence (\bar{x}_n) converges along N to x , prove that

$$\frac{x}{r} \leq \liminf \bar{x}_n \leq \limsup \bar{x}_n \leq r x .$$

8. Suppose that if you are s minutes early for an appointment then you incur the cost cs , and if you are s minutes late then you incur the cost ks , where c and k are finite positive constants. Suppose that the travel time from where you presently are to the location of your appointment is a random variable having continuous probability density function f , distribution function F , and finite expectation. Determine the time at which you should depart if you want to minimize your expected cost.

9. For each $n \geq 1$, let the function $g_n : [0, \infty) \rightarrow [0, \infty)$ be defined by

$$g_n(x) = \begin{cases} nx, & 0 \leq x \leq \frac{1}{n} \\ \frac{1}{nx}, & \frac{1}{n} < x < \infty. \end{cases}$$

- (a) Show that $\lim_n g_n(x) = 0$ for all $x \geq 0$.
(b) Is the convergence uniform in x ? Prove your answer.

10. A parallel system functions if and only if at least one of its components is functioning. A device consists of n components in parallel. The lifetimes L_1, \dots, L_n of the components are independent and exponentially distributed with the same rate λ . Let L be the lifetime of the system.

(a) Compute

$$P\{L \leq t\}, \quad t \geq 0.$$

(b) Show that L can be represented as

$$L = W_1 + \dots + W_n,$$

where W_1, \dots, W_n are independent exponential variables with respective parameters $n\lambda, (n-1)\lambda, \dots, \lambda$.

11. Let A_1, A_2, \dots, A_n be arbitrary events, and define C_k to be the event that at least k of these events occur. Show that

$$\sum_{k=1}^n P(C_k) = \sum_{k=1}^n P(A_k).$$

HINT: Define the random variable X as the number of A_1, \dots, A_n that occur. How are the two quantities related to X ?

12. Let A be an $n \times n$ nonsingular matrix, and let B denote its classical adjoint. That is, $B = C^T$, where $C = [c_{ij}]$ is the matrix of cofactors:

$$c_{ij} = (-1)^{i+j} \det A_{ij},$$

where A_{ij} is the submatrix of A obtained by deleting the i th row and j th column. Prove that $\det B = (\det A)^{n-1}$.

13. Let $B = \begin{bmatrix} 1 & x \\ x & 3 \end{bmatrix}$ where x represents a real number. For what values of x does there exist a real symmetric matrix A such that $A^2 = B$?

14. Let $p(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0$ be a polynomial with real coefficients, and let M denote its companion matrix

$$M = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -c_0 & -c_1 & -c_2 & \cdots & -c_{n-2} & -c_{n-1} \end{bmatrix}.$$

Let $\lambda_1, \dots, \lambda_n$ denote the roots of p counted with multiplicity, and define the Vandermonde matrix V to be

$$V := \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_{n-1} & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \cdots & \lambda_{n-1}^2 & \lambda_n^2 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \lambda_3^{n-1} & \cdots & \lambda_{n-1}^{n-1} & \lambda_n^{n-1} \end{bmatrix}.$$

Show that $MV = VD$, where D is the diagonal matrix whose diagonal entries are (in order) $\lambda_1, \dots, \lambda_n$.

15. Suppose that X_1, X_2, X_3, \dots is a sequence of nonnegative random variables, and define $M_n := \max_{k \leq n} X_k$.

Throughout the problem (and your solution), we agree to use the following shorthand for expectation, where Y is a random variable and A is an event (and I_A is the indicator of A):

$$E(Y; A) := E(YI_A).$$

We also conveniently drop $\{ \}$ as the following example illustrates:

$$E(Y; Y > y) := E(Y; \{Y > y\}) = E(YI_{\{Y > y\}}).$$

- (a) Show for all $\alpha > 0$ that

$$E(M_n; M_n > \alpha) \leq \sum_{k=1}^n E(X_k; X_k > \alpha).$$

- (b) If X_1, X_2, X_3, \dots are supposed further to be identically distributed (but not necessarily independent) with finite expectation, use part (a) to show that

$$\lim_{n \rightarrow \infty} \frac{1}{n} E(M_n) = 0.$$