Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Fall Session

Wednesday, August 26, 2009

Instructions: Read carefully!

1. This **closed-book** examination consists of 15 problems, each worth 5 points. The passing grade has been set at 50 points, i.e. 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been grouped by topic, but there are roughly equally many mainly motivated by each of the three areas identified in the syllabus (linear algebra; real analysis; probability;). Nor have the problems been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. The examination will begin at 8:30 AM; lunch and refreshments will be provided. The exam will end just before 5:00 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. In answering a question on a multiple-choice test where each question has m answers from which to choose, a student either knows (and hence chooses) the correct answer or doesn't know and guesses (uniformly) randomly. Let p denote the probability that the student knows the answer. What is the conditional probability that the student knows the answer to a question given that she answers it correctly?

Solution:

This is a Bayes'-theorem Example in Chapter 3 of Ross. Let K and C denote, respectively, the event that the student knows the correct answer and the event that the student answers the question correctly. Then the given information is that $P(K) = p$, $P(C|K) = 1$, and $P(C|K^c) = 1/m$. Therefore the desired conditional probability is

$$\begin{aligned} P(K|C) &= \frac{P(C \cap K)}{P(C)} = \frac{P(K)P(C|K)}{P(K)P(C|K) + P(K^c)P(C|K^c)} \\ &= \frac{p \times 1}{(p \times 1) + (1-p)(1/m)} = \frac{mp}{1 + (m-1)p}. \end{aligned}$$

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2. Let A be an $n \times n$ positive definite matrix and let B be an $n \times m$ matrix with linearly independent columns. Show that the following matrix is nonsingular:

$$\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix},$$

where B^* is the conjugate transpose of B.

Solution:

Suppose that

$$\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0.$$

Then we have $Ax + By = 0$ and $B^*x = 0$. We want to show $x = 0$ and $y = 0$. Multiplying the first equation by x^* and using the second equation, we get

$$x^*Ax + x^*By = x^*Ax = 0.$$

It follows from the positive definiteness of A that $x = 0$. Then the first equation is reduced to $By = 0$. Hence we also have $y = 0$, because B has linearly independent columns.

3. Consider the sequence of continuous functions on $[0,1]$

$$f_n(x) = \frac{x^2}{x^2 + (nx-1)^2},$$

for $n = 1, 2, 3, ...$ Show that the sequence is uniformly bounded and that

$$\lim_{n\to\infty} f_n(x) = 0$$

pointwise, but that no subsequence $f_{n_k}(x)$ exists which converges uniformly on $[0,1]$.

Solution:

Clearly $|f_n(x)| \leq 1$ for all x, n, so that the sequence is uniformly bounded on $[0,1]$. Also, for each $x > 0$, $(nx-1)^2 \to \infty$ as $n \to \infty$, so that

$$\lim_{n\to\infty} f_n(x) = 0$$

when $x > 0$. Since $f_n(0) = 0$, pointwise convergence to 0 holds for all $x \in [0,1]$. However,

$$f_n(1/n) = 1$$

for all n, so that there can be no subsequence which converges to 0 uniformly on $[0,1]$. This example lacks the property of *equicontinuity*, which, together with pointwise boundedness, would guarantee the existence of a uniformly convergent subsequence.

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4. Determine whether each of the following sums converges or diverges:

$$\sum_{n=2}^{\infty} (\ln n)^{-\ln n}, \quad \sum_{n=2}^{\infty} e^{-(\ln n)^2}.$$

Solution:

The first series converges by a comparison test since $(\ln n)^{-\ln n} < n^{-2}$ for all n such that $\ln \ln n > 2$, and hence for all n sufficiently large. The second series converges because $e^{-(\ln n)^2} \leq e^{-2\ln n} = n^{-2}$ for all n sufficiently large.

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5. Let p be the probability of 'head' for a biased coin. Let q_n be the probability of obtaining an even number of heads after n tosses. Find two numbers c and α (depending on p) such that $q_n - 1/2 = c\alpha^n$.

   Hint: compute the difference between the probability of obtaining an even number of heads and the probability of obtaining an odd number of heads.

   Solution:

   The probability is

$$q_n = \sum_{0 \le 2k \le n} \binom{n}{2k} p^{2k}(1-p)^{n-2k}.$$

   We also have

$$1 - q_n = \sum_{1 \le 2k+1 \le n} \binom{n}{2k+1} p^{2k+1}(1-p)^{n-2k-1}.$$

   and taking the difference

$$\begin{aligned} 2q_n - 1 &= \sum_{0 \le 2k \le n} \binom{n}{2k} p^{2k}(1-p)^{n-2k} \\ &\quad - \sum_{1 \le 2k+1 \le n} \binom{n}{2k+1} p^{2k+1}(1-p)^{n-2k-1} \\ &= \sum_{0 \le k \le n} \binom{n}{k} (-p)^k (1-p)^{n-k} \\ &= (1-2p)^n. \end{aligned}$$

   We therefore get $q_n = (1 + (1-2p)^n)/2$: $c = 1/2$ and $\alpha = 1 - 2p$.

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6. Give necessary and sufficient conditions on the constant c for the matrix

$$A = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ c & 0 & 1 \end{bmatrix}$$

   to be positive definite.

   Hint: It will help to show that 1 is an eigenvalue of the matrix no matter what the value of c is.

Solution:

Taking

$$v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

we see that $Av = v$. A quick calculation shows that the characteristic polynomial of A takes the form

$$p(x) = x^3 - 3x^2 + 3x - 1 - c^2x + c^2.$$

By the above observation, we know that $x - 1$ divides this polynomial, and we see that

$$p(x)/(x-1) = x^2 - 2x + 1 - c^2 = (x-1)^2 - c^2$$

so that

$$p(x) = (x-1)(x-1-c)(x-1+c).$$

Therefore, the eigenvalues are 1, $1 + c$ and $1 - c$, and for the three to be positive it is necessary and sufficient that $|c| < 1$.

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7. With a proof, determine which of e^π and π^e is larger.

Solution:

Taking natural logarithms, and using the fact that the natural logarithm function ln is increasing,

$$e^\pi > \pi^e \quad \text{if and only if} \quad \pi \ln e > e \ln \pi \quad \text{if and only if} \quad (\ln e)/e > (\ln \pi)/\pi.$$

To prove this last inequality [call it $(*)$], let $f(x) := (\ln x)/x$ for $x > 0$. Straightforward differentiation gives

$$f'(x) = \frac{(1/x)x - (1)\ln x}{x^2},$$

which has the same sign as $1 - \ln x$ and so is positive [i.e., $f(x)$ is increasing] for $0 < x < e$ and negative [i.e., $f(x)$ is decreasing] for $x > e$. Thus f is uniquely maximized at $x = e$. So $f(e) > f(\pi)$, which is the desired result $(*)$.

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8. Determine, with justification, whether the following matrices are diagonalizable.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Solution:

Since 1,2 and 3 are three distinct roots of the characteristic polynomial of B, this matrix is diagonalizable.

For A, we need to compute the dimension of the space of eigenvectors for the eigenvalue 2, which must have dimension 2, since 2 is a double root of the characteristic polynomial. Solving $AX = 2X$ with $X^T = (x, y, z)$ gives $x = 2y$ and $z = 0$, which defines a line and A is not diagonalizable.

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9. Show that if $P(E_i) = 1$ for $i = 1, 2, 3, \ldots$, then $P(\cap_{i=1}^{\infty} E_i) = 1$.

Solution:

Consider the complementary event.

By DeMorgan's Law,

$$(\cap_{i=1}^{\infty} E_i)^c = \cup_{i=1}^{\infty} E_i^c.$$

By Boole's Inequality,

$$P\left(\cup_{i=1}^{\infty} E_i^c\right) \leq \sum_{i=1}^{\infty} P(E_i^c)$$

$$= \sum_{i=1}^{\infty} [1 - P(E_i)]$$

by complementation, and

$$= 0,$$

by the hypothesis.

Thus,

$$P\left(\cap_{i=1}^{\infty} E_i\right) = 1 - P\left(\cup_{i=1}^{\infty} E_i^c\right) = 1.$$

10. Let U be a random variable uniformly distributed over $(0,1)$. Suppose that, conditionally given $U = u$, the random variables $X_1, X_2, \dots$ are independent and identically distributed Bernoulli random variables with success probability u. For $0 \le s \le n$, find

$$P(X_{n+1} = 1 \mid X_1 + \cdots + X_n = s)$$

in simplest form.

Hint: You may use without proof the identity

$$\int_0^1 x^{a-1}(1-x)^{b-1}\,dx = \frac{(a-1)!(b-1)!}{(a+b-1)!},$$

valid for integers $a \ge 1$ and $b \ge 1$.

Solution:

Solution #1: This is Laplace's rule of succession; see, for example,

`http://en.wikipedia.org/wiki/Rule of succession` .

Let p denote the conditional probability in question. Then, making two uses of the identity in the hint,

$$\begin{aligned} p &= \frac{P(X_1 + \cdots + X_n = s,\ X_{n+1} = 1)}{P(X_1 + \cdots + X_n = s)} = \frac{\int_0^1 \binom{n}{s} u^{s+1}(1-u)^{n-s}\,du}{\int_0^1 \binom{n}{s} u^{s}(1-u)^{n-s}\,du} \\ &= \frac{\binom{n}{s}\frac{(s+1)!(n-s)!}{(n+2)!}}{\binom{n}{s}\frac{s!(n-s)!}{(n+1)!}} = \frac{s+1}{n+2}. \end{aligned}$$

Solution #2: This variant of the first solution might be utilized by students with knowledge of Bayesian statistics and the beta distribution. Using Bayes' theorem, we see that the posterior distribution of U given $X_1 + \cdots + X_n = s$ is $\mathrm{Beta}(s+1, n+1-s)$. Therefore, the conditional probability in question is

$$\begin{aligned} p &= \int_0^1 P(X_{n+1} = 1 \mid U = u, X_1 + \cdots + X_n = s)\frac{(n+1)!}{s!(n-s)!}u^s(1-u)^{n-s}\,du \\ &= \int_0^1 \frac{(n+1)!}{s!(n-s)!}u^{s+1}(1-u)^{n-s}\,du = \frac{(n+1)!}{s!(n-s)!}\,\frac{(s+1)!(n-s)!}{(n+2)!} = \frac{s+1}{n+2}. \end{aligned}$$

11. Define the sequence $(T_n)_{n=1,2,\dots}$ by

$$T_n = \sum_{r=1}^{n} \frac{1}{r} - \ln n.$$

Show that (T_n) is decreasing, and tends to a limit in $[0, 1]$.

Hint: Verify and use the inequality

$$\frac{1}{n+1} \leq \int_n^{n+1} \frac{1}{x} dx, \quad \text{for } n \geq 1.$$

Solution:

Observe that $1/x \geq 1/(n+1)$ for $x \in [n, n+1]$, and then integrate to get inequality. The sequence (T_n) is decreasing because

$$T_{n+1} - T_n = \frac{1}{n+1} - \int_n^{n+1} \frac{1}{x} dx \leq 0.$$

Note that $T_1 = 1$, and

$$T_n \geq \sum_{r=1}^{n-1} \left(\frac{1}{r} - \int_r^{r+1} \frac{1}{x} dx \right) \geq 0.$$

A decreasing bounded sequence converges to a limit.

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12. Let A be an $n \times n$ real, symmetric, positive definite matrix, and b be a vector in $\mathbb{R}^n$. Define a real-valued function

$$f : \mathbb{R}^n \to \mathbb{R} : x \mapsto f(x) = \frac{1}{2} x^T A x - b^T x.$$

Show that the unique minimum of $f(x)$ is given by solving $Ax = b$ for $x = A^{-1}b$.

Solution:

Since A is invertible (why?), $A^{-1}b$ is uniquely determined. Let $x_0 = A^{-1}b$. Then $b = Ax_0$ and $f(x_0) = \frac{1}{2}x_0^T A x_0 - b^T x_0 = \frac{1}{2}x_0^T A x_0 - x_0^T A x_0 = -\frac{1}{2}x_0^T A x_0$. Thus

$$f(x) - f(x_0) = \frac{1}{2} x^T A x - x_0^T A x + \frac{1}{2} x_0^T A x_0 = \frac{1}{2}(x - x_0)^T A (x - x_0), \quad \forall x \in \mathbb{R}^n.$$

Since A is positive definite, $f(x) - f(x_0) > 0$ for all $x \neq x_0$. Thus f has the unique minimum at $x_0 = A^{-1}b$.

13. Let X_1 and X_2 be independent random variables, each exponentially distributed with parameter 1. Define $M := \max\{X_1, X_2\}$ and $S := \frac{1}{2}X_1 + X_2$. Prove that M and S have the same distribution.

Solution:

Solution #1: There are many possible solutions. Here is perhaps the simplest. As is well known and easily established, the minimum of n independent exponential random variables with parameters $\lambda_1, \ldots, \lambda_n$ has an exponential distribution with parameter equal to the sum $\lambda_1 + \cdots + \lambda_n$ of the parameters. Thus $m := \min\{X_1, X_2\}$ has the exponential distribution with parameter 2, i.e., the same distribution as $\frac{1}{2}X_1$; and, by symmetry and the memoryless property of the exponential distribution, the difference $M - m$ and m are independent random variables, with $M - m$ having the exponential distribution with parameter 1, i.e., the same distribution as X_2.

Solution #2: The random variable M has distribution function

$$P(M \leq t) = P(X_1 \leq t)P(X_2 \leq t) = (1 - e^{-t})^2 = 1 - 2e^{-t} + e^{-2t}, \quad t \geq 0,$$

and so density function f (say) given by

$$f(t) = 2(e^{-t} - e^{-2t}), \quad t \geq 0,$$

and moment generating function ϕ (say) given by

$$\begin{aligned} \phi(z) &= 2\left(\frac{1}{1-z} - \frac{1}{2-z}\right) = \frac{2}{(1-z)(2-z)} = \frac{1}{1-(z/2)} \times \frac{1}{1-z} \\ &= Ee^{z(X_1/2)}Ee^{zX_2} = Ee^{zS}, \quad -\infty < z < 1. \end{aligned}$$

Since M and S have the same moment generating function ϕ and ϕ is finite in the open interval $(-\infty, 1)$ containing the origin, the desired result follows.

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14. Let $\mathcal{V}$ be a finite-dimensional vector space. For a linear operator from $\mathcal{V}$ to itself, $L : \mathcal{V} \to \mathcal{V}$, a number λ is called an *eigenvalue* of the operator L if there exists a vector $v \in \mathcal{V}$, $v \neq 0$, such that $Lv = \lambda v$. Let L be an *idempotent* linear operator from $\mathcal{V}$ to itself. This means that $L \circ L = L$. Find all possible eigenvalues of such an operator.

Solution:

Let v be an (nonzero) eigenvector and λ is corresponding eigenvalue of L. Then

$$\begin{aligned} L(v) &= \lambda v \\ L[L(v)] &= L[\lambda v] = \lambda^2 v \\ \text{but also } L[L(v)] &= (L \circ L)(v) = L(v) = \lambda v \end{aligned}$$

and so $\lambda^2 v = \lambda v$ which yields $\lambda^2 = \lambda$. From this we have $\lambda = 0$ or $\lambda = 1$.

The all zeros matrix and the identity matrix show that both of these eigenvalues are achievable.

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15. Suppose $f : [0, \infty) \to \mathbb{R}$ is a continuous function that satisfies the following equation for every $x \geq 0$:

$$f(x) + \int_0^x f(y)\,dy = 1.$$

Show that any such f must be continuously differentiable, and find f.

Solution:

First of all, any such f must satisfy $f(0) = 1$. Furthermore, for any $x > 0$ and h for which $x + h > 0$,

$$\frac{f(x+h) - f(x)}{h} + \frac{1}{h} \int_x^{x+h} f(y)\,dy = 0.$$

Since f is continuous, $\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(y)\,dy = f(x)$, which in turn implies the existence of the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x), \quad \forall x > 0.$$

Therefore, the function f must satisfy the first-order initial value problem:

$$f'(x) + f(x) = 0 \ \ (x > 0), \quad f(0) = 1.$$

The solution of this equation is $f(x) = e^{-x}$.
