

Department of Applied Mathematics and Statistics
The Johns Hopkins University

INTRODUCTORY EXAMINATION—FALL SESSION

Wednesday, August 26, 2009

Instructions: Read carefully!

1. This **closed-book** examination consists of 15 problems, each worth 5 points. The passing grade has been set at 50 points, i.e. $2/3$ of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.
3. The problems have not been grouped by topic, but there are roughly equally many mainly motivated by each of the three areas identified in the syllabus (linear algebra; real analysis; probability;). Nor have the problems been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
4. Start your answer to each problem on a **NEW** sheet of paper. Write only on **ONE SIDE** of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your **NAME** and the **PROBLEM NUMBER** on each sheet.
5. The examination will begin at 8:30 AM; lunch and refreshments will be provided. The exam will end just before 5:00 PM. You may leave before then, but in that case you may not return.
6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
7. **No calculators of any sort are needed or permitted.**

1. In answering a question on a multiple-choice test where each question has m answers from which to choose, a student either knows (and hence chooses) the correct answer or doesn't know and guesses (uniformly) randomly. Let p denote the probability that the student knows the answer. What is the conditional probability that the student knows the answer to a question given that she answers it correctly?
2. Let A be an $n \times n$ positive definite matrix and let B be an $n \times m$ matrix with linearly independent columns. Show that the following matrix is nonsingular:

$$\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix},$$

where B^* is the conjugate transpose of B .

3. Consider the sequence of continuous functions on $[0, 1]$

$$f_n(x) = \frac{x^2}{x^2 + (nx - 1)^2},$$

for $n = 1, 2, 3, \dots$. Show that the sequence is uniformly bounded and that

$$\lim_{n \rightarrow \infty} f_n(x) = 0$$

pointwise, but that no subsequence $f_{n_k}(x)$ exists which converges uniformly on $[0, 1]$.

4. Determine whether each of the following sums converges or diverges:

$$\sum_{n=2}^{\infty} (\ln n)^{-\ln n}, \quad \sum_{n=2}^{\infty} e^{-(\ln n)^2}.$$

5. Let p be the probability of 'head' for a biased coin. Let q_n be the probability of obtaining an even number of heads after n tosses. Find two numbers c and α (depending on p) such that $q_n - 1/2 = c\alpha^n$.

Hint: compute the difference between the probability of obtaining an even number of heads and the probability of obtaining an odd number of heads.

6. Give necessary and sufficient conditions on the constant c for the matrix

$$A = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ c & 0 & 1 \end{bmatrix}$$

to be positive definite.

Hint: It will help to show that 1 is an eigenvalue of the matrix no matter what the value of c is.

7. With a proof, determine which of e^π and π^e is larger.

8. Determine, with justification, whether the following matrices are diagonalizable.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

9. Show that if $P(E_i) = 1$ for $i = 1, 2, 3, \dots$, then $P(\cap_{i=1}^{\infty} E_i) = 1$.

10. Let U be a random variable uniformly distributed over $(0, 1)$. Suppose that, conditionally given $U = u$, the random variables X_1, X_2, \dots are independent and identically distributed Bernoulli random variables with success probability u . For $0 \leq s \leq n$, find

$$P(X_{n+1} = 1 \mid X_1 + \dots + X_n = s)$$

in simplest form.

Hint: You may use without proof the identity

$$\int_0^1 x^{a-1}(1-x)^{b-1} dx = \frac{(a-1)!(b-1)!}{(a+b-1)!},$$

valid for integers $a \geq 1$ and $b \geq 1$.

11. Define the sequence $(T_n)_{n=1,2,\dots}$ by

$$T_n = \sum_{r=1}^n \frac{1}{r} - \ln n.$$

Show that (T_n) is decreasing, and tends to a limit in $[0, 1]$.

Hint: Verify and use the inequality

$$\frac{1}{n+1} \leq \int_n^{n+1} \frac{1}{x} dx, \quad \text{for } n \geq 1.$$

12. Let A be an $n \times n$ real, symmetric, positive definite matrix, and b be a vector in \mathbb{R}^n . Define a real-valued function

$$f : \mathbb{R}^n \rightarrow \mathbb{R} : x \mapsto f(x) = \frac{1}{2}x^T Ax - b^T x.$$

Show that the unique minimum of $f(x)$ is given by solving $Ax = b$ for $x = A^{-1}b$.

13. Let X_1 and X_2 be independent random variables, each exponentially distributed with parameter 1. Define $M := \max\{X_1, X_2\}$ and $S := \frac{1}{2}X_1 + X_2$. Prove that M and S have the same distribution.

14. Let \mathcal{V} be a finite-dimensional vector space. For a linear operator from \mathcal{V} to itself, $L : \mathcal{V} \rightarrow \mathcal{V}$, a number λ is called an *eigenvalue* of the operator L if there exists a vector $v \in \mathcal{V}$, $v \neq 0$, such that $Lv = \lambda v$. Let L be an *idempotent* linear operator from \mathcal{V} to itself. This means that $L \circ L = L$. Find all possible eigenvalues of such an operator.

15. Suppose $f : [0, \infty) \rightarrow \mathbb{R}$ is a continuous function that satisfies the following equation for every $x \geq 0$:

$$f(x) + \int_0^x f(y) dy = 1.$$

Show that any such f must be continuously differentiable, and find f .