Department of Applied Mathematics and Statistics
The Johns Hopkins University

INTRODUCTORY EXAMINATION–FALL SESSION

August 27, 2008

Instructions: Read carefully!

1. This **closed-book** examination consists of 15 problems, each worth 5 points. The passing grade has been set at 50 points, i.e. 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been grouped by topic, but there are roughly equally many mainly motivated by each of the three areas identified in the syllabus (linear algebra; real analysis; probability;). Nor have the problems been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. The examination will begin at 8:30 AM; lunch and refreshments will be provided. The exam will end just before 5:00 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. The Legendre transform of a convex function $F(\mathbf{x})$ on $\mathbb{R}^n$ is another convex function on $\mathbb{R}^n$ defined by

$$F^*(\mathbf{y}) = \sup_{\mathbf{x}}[(\mathbf{x},\mathbf{y}) - F(\mathbf{x})],$$

where $(\mathbf{x},\mathbf{y})$ is the standard Euclidean inner product. If $Q_A(\mathbf{x}) = \frac{1}{2}(\mathbf{x},\mathbf{A}\mathbf{x})$ is a quadratic form defined by a non-singular real, symmetric $n \times n$ matrix $\mathbf{A}$, calculate $Q_A^*(\mathbf{y})$.

2. Find the inverse of the $n \times n$ matrix $A := J - I$, where J is the matrix with every entry equal to 1 and I is the identity matrix.

3. Fix an integer $r \geq 1$. Let X be a random variable with the Gamma$(r,1)$ distribution, i.e., with density

$$f_X(x) = \frac{1}{(r-1)!} e^{-x} x^{r-1}, \quad x > 0.$$

   (a) Calculate the moment generating function M of X. For which values of $t \in \mathbb{R}$ do we have $M(t) < \infty$?

   (b) For a real number $a > r$, derive the best Chernoff bound on the tail probability $P\{X \geq a\}$. [REMINDER: "Chernoff bound" refers to a bound on $P\{e^{tX} \geq e^{ta}\}$ for some $t > 0$ obtained using Markov's inequality.]

4. The characteristic function of a random variable X is

$$\phi_X(t) = E(e^{itX}), \ -\infty < t < \infty.$$

   (a) Compute $\phi_X(t)$ if X has the uniform distribution on $[0,1]$.

   (b) Identify a random variable with characteristic function

$$\phi(t) = \frac{2(1-\cos t)}{t^2}$$

   by considering two independent random variables, each uniform on $[0,1]$.

5. Let $\{a_1, a_2, \ldots\}$ be a sequence of real numbers for which

$$\lim_{n\to\infty} a_n = 0.$$

   Prove that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} a_k = 0.$$

6. Suppose that A is a $n \times n$ real matrix such that both A and $I + A$ are nonsingular. Show that

   (a) $(I+A)^{-1} = I - (A^{-1}+I)^{-1}$.
   (b) $\mathrm{trace}(I+A)^{-1} + \mathrm{trace}(A^{-1}+I)^{-1} = n$.

7. Show that for any rearrangement $b_1, \cdots, b_n$ of the positive numbers $a_1, \cdots, a_n$,

$$\frac{1}{n}\left(\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n}\right) \geq 1.$$

8. Consider a sequence $x^{(n)},\ n = 1, 2, \dots$ of elements of $\mathbb{R}^d$ with the following properties:

   (i) $\lim_{n\to\infty}(x_i^{(n)} - x_j^{(n)}) = 0$ for all $1 \leq i, j \leq d$,
   (ii) $\max_{1\leq i\leq d} x_i^{(n)},\ n = 1, 2, \dots,$ is a nonincreasing sequence, and
   (iii) $\min_{1\leq i\leq d} x_i^{(n)},\ n = 1, 2, \dots,$ is a nondecreasing sequence.

   (Here $x_i^{(n)}$ denotes the i-th coordinate of $x^{(n)}$.) Prove that $x_i^{(n)},\ n = 1, 2, \dots$ converges to the same limit for $i = 1, \dots, d$

9. Let X_1 and X_2 be independent and identically distributed $Uniform(0,1)$ random variables. Let $Y = \min(X_1, X_2)$ and $Z = \max(X_1, X_2)$. Give the conditional density of Z given $Y = y$.

10. Prove that there exists an $n \times n$ real matrix A with $A^2 = -I$ if and only if n is even.

11. If X is a Poisson random variable with parameter λ, where $0 < \lambda < 1$, find $E[X!]$.

12. Let V be the vector space of real polynomials with degree 2 or less. Consider the dot product on V

$$P \cdot Q = -\int_0^1 P(x)Q(x)\ln x dx.$$

(a) Prove that this dot product is well defined and provides a positive definite linear form on V.

(b) If the polynomial $Q = ax + b$ is orthogonal to $P \equiv 1$ what linear relation do the constants a and b satisfy?

13. For positive integers n, m with $n < m$ define

$$f(n,m) = \frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots + \frac{1}{m^2}.$$

Prove that for every positive real number ε there must exist an integer T so that if $T \leq n < m$ then $f(n,m) < \varepsilon$.

14. Let f be a nonnegative continuous function with $\int_{x=0}^{\infty} f(x)dx < +\infty$. Prove that

$$\lim_{n\to\infty} \frac{1}{n} \int_{x=0}^{n} x f(x)dx = 0.$$

Hint: Write the integral as a double-integral.

15. The lifetime of a lightbulb has an exponential distribution with parameter 1. If n lightbulbs have independent lifetimes and all are turned on simultaneously, what is the expected value of the time at which the first bulb fails?