Department of Applied Mathematics and Statistics
The Johns Hopkins University

INTRODUCTORY EXAMINATION–FALL SESSION

Wednesday, August 30, 2006

Instructions: Read carefully!

1. This **closed-book** examination consists of 20 problems (sorry, no choices), each worth 5 points. The passing grade has been set at $66\frac{2}{3}\%$. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.
3. The problems have not been grouped by topic, but there are roughly equally many mainly motivated by each of the four areas identified in the syllabus (linear algebra; real analysis; probability; discrete mathematics and operations research/optimization). Nor have the problems been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.
5. The examination will begin at 8:30 AM; lunch and refreshments will be provided. The exam will end just before 5:00 PM. You may leave before then, but in that case you may not return.
6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
7. **No calculators of any sort are needed or permitted.**

1. An $n \times n$ matrix $P = (p_{ij})$ is said to be *stochastic* if all of its entries are nonnegative and the sum of the entries in each row is 1.

   If P is an $n \times n$ stochastic matrix, prove the existence of a nonzero nonnegative solution to the system of equations
$$\sum_{i=1}^{n} y_i p_{ij} = y_j, \ j = 1, \dots, n.$$
   (Hint: One appproach involves introducing a related linear program.)

2. Which (if either) of these sums converges?
$$\sum_{n=2}^{\infty} \frac{1}{n \log n} \qquad \text{and} \qquad \sum_{n=2}^{\infty} \frac{1}{n \log^2 n}$$
   Justify your answer.

   For this problem, we assume the logarithms are base e, but does that affect your answer?

3. Let O be an $n \times n$ real orthogonal matrix, i.e. such that
$$O^T O = OO^T = I.$$
   (a) Prove that $\det O = \pm 1$.

   (b) If n is odd, prove that $O\mathbf{x} = (\det O)\mathbf{x}$ for some nonzero $\mathbf{x} \in \mathbb{R}^n$.

4. Let G be a simple graph (no loops or multiple edges) with vertex set $\{v_1, v_2, v_3, v_4, v_5\}$. The degree sequence of G is a list of the degrees of the vertices in the graph, i.e., $d(v_1), d(v_2), d(v_3), d(v_4), d(v_5)$, often listed in numerical order.

   Only one of the following three sequences can possibly be the degree sequence of G.

   (a) 1, 2, 2, 3, 3

   (b) 1, 1, 3, 3, 4

   (c) 2, 2, 2, 3, 3

   Prove that two of these sequences cannot be the degree sequence of G and then demonstrate that the third sequence is feasible by drawing a picture of G.

5. Let X be a 2×4 real matrix. We calculate

$$X^T X = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 4 & 2 & 4 \\ 1 & 2 & 2 & 3 \\ 1 & 4 & 3 & ? \end{bmatrix}.$$

Find (with proof) the missing entry (denoted with a question mark).

6. Here $\{f_n : \ n = 1, 2, \ldots\}$ is a sequence of continuous real-valued functions defined on a common interval $I = [a,b]$ such that, for each $x \in I$, the sequence $f_n(x)$ converges to a limit, denoted $f(x)$. Prove that if the convergence is uniform, then the limit-function f is also continuous.

7. Let A be an $n \times n$ matrix and assume that A has n distinct eigenvalues. Let $\mathscr{V}$ be the set of all $n \times n$ matrices B such that $AB = BA$. Show that $\mathscr{V}$ is a vector space and compute its dimension. (Hint: Show that B maps eigenvectors of A to eigenvectors of A.)

8. Let $a(n,x) = \lfloor \frac{n}{2} + \frac{x}{2}\sqrt{n} \rfloor$ where n is a positive integer, x is real and $\lfloor y \rfloor$ denotes the greatest integer less than or equal to y. Evaluate

$$\lim_{n \to \infty} 2^{-n} \sum_{k=0}^{a(n,x)} \binom{n}{k}.$$

[Hint: Use the Central Limit Theorem.]

9. The number of births per day in a small town hospital has the following distribution:

# births	0	1	2	3	4
probability	.25	.45	.14	.11	.05

Assume that each baby has probability 1/2 to be a girl. What is the most likely number of births in a day if it is known that exactly two girls are born ?

10. Let $a_1, a_2, \ldots, a_n$ be positive numbers. Prove that

$$\sum_{i=1}^{n} a_i \sum_{j=1}^{n} 1/a_j \geq n^2$$

and equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

11. Let X have distribution function F_X, with probability density function f_X.
Let Y have distribution function F_Y, with probability density function f_Y.

Recall that $X <^{st} Y$ (read "X is stochastically smaller than Y") means that $F_X(z) \geq F_Y(z)$ for all z with strict inequality for at least one z.

Prove that $X <^{st} Y \implies P[X < Y] > 1/2$.

12. (a) For real numbers $x_1, \ldots, x_n$, find the maximum of

$$f(x_1, x_2, \ldots, x_n) := (x_1 x_2 \cdots x_n)^2,$$

subject to the constraint

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 1.$$

Do *not* invoke the result of part (b).

(b) Show that the geometric mean of a collection of nonnegative real numbers $\{a_1, \ldots, a_n\}$ does not exceed their arithmetic mean; that is,

$$(a_1 a_2 \cdots a_n)^{1/n} \leq \frac{1}{n}(a_1 + a_2 + \cdots + a_n).$$

13. Let $\hat{x}$ be a feasible point to the following linear program:

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & a_i^T x \geq \beta_i \quad i = 1, \ldots, m. \end{array}$$

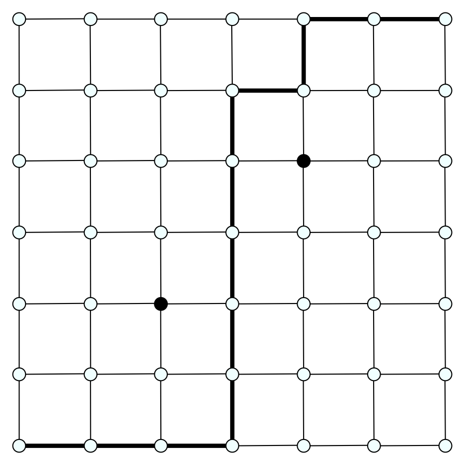
Let $I(\hat{x}) = \{i : a_i^T \hat{x} = \beta_i\}$. Show that if

$$c = \sum_{i \in I(\hat{x})} \lambda_i a_i \quad \text{and} \quad \lambda_i \geq 0,$$

then $\hat{x}$ is optimal.

14. Let $B(r)$ denote the ball of radius r centered at the origin (i.e., the set of points whose Euclidean distance from the origin is less than or equal to r) and let X be a three-dimensional random vector uniformly distributed over $B(r)$ (i.e., the result of choosing a point in $B(r)$ at random). Find the mean of $\|X\|$, the distance from X to the origin.

15. Consider the graph in the accompanying figure. There are many shortest paths from the lower left corner to the upper right corner. How many of these avoid the two vertices that are colored black? One such path is highlighted.



16. Given n numbers $x_1, \ldots, x_n$, the *Vandermonde matrix* $V = V(x_1, \ldots, x_n)$ is, by definition, the matrix $V = [v_{ij}]$ with

$$v_{ij} = x_j^{i-1}.$$

(a) For $n = 2$, verify that

$$\det V = \prod_{1 \leq i < j \leq n} (x_j - x_i).$$

(b) Prove the formula in part (a) for all $n \geq 1$. [HINT: The solution is simpler if you do *not* use induction or any explicit Laplace expansion of the determinant, but rather think about the determinant as a polynomial.]

17. Determine whether or not the vector x=(1,0,1,0) is an optimal solution of the following linear program:

$$\begin{array}{lll} \min & -x_1+2x_2-x_3-x_4 & \\ \text{s.t.} & x_1+x_2-x_3+2x_4 & \geq -2 \\ & x_1+2x_2-x_3+x_4 & = 0 \\ & -x_1-x_2-x_3-x_4 & \geq -2 \\ & x_1,x_2,x_3 \geq 0 & x_4 \quad \text{unrestricted.} \end{array}$$

18. A primitive model for heat conduction leads to the equation

$$\begin{bmatrix} u_n \\ v_n \end{bmatrix} = A^n \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}, n = 1,2,\cdots, A = \begin{bmatrix} 7/9 & 1/9 \\ 1/9 & 7/9 \end{bmatrix},$$

where u_n and v_n represent temperatures at times $n = 0,1,\ldots$. Find the limits c_u and c_v, where $u_n \to c_u$ and $v_n \to c_v$ as $n \to \infty$.
Proceed as follows:
(a) Find the eigenvalues of A.
(b) Represent A^n as CDC^{-1}, where D is a diagonal matrix.
(c) Use this representation of A^n to find the limits c_u and c_v.

19. For a set of randomly chosen people, let $E_{i,j}$ denote the event that persons i and j have the same birthday. (Assume that each person is equally likely to have any of the 365 days of the year as his or her birthday, and that different persons' birthdays are independent.)

(a) Find $P[E_{3,4}|E_{1,2}]$.

(b) Find $P[E_{1,3}|E_{1,2}]$.

(c) Find $P[E_{2,3}|E_{1,2} \cap E_{1,3}]$.

(d) What can you conclude from the previous three parts about the independence of the events $\{E_{i,j}\}$?

20. If $\{x_n\}$ is a sequence of real numbers, then its *limit infimum* is defined by

$$x_* = \lim_{n\to\infty} \inf_{k\geq n} x_k$$

Given such a sequence, prove that there is a subsequence $\{x_{k_n}\}$ such that $\lim_{n\to\infty} x_{k_n} = x_*$ but no subsequence such that $\lim_{n\to\infty} x_{k_n} = y_* < x_*$.