Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Fall session

Wednesday, August 31st, 2005

Instructions: Read carefully!

1. This **closed-book** examination consists of 20 problems (sorry, no choices), each worth 5 points. The passing grade has been set at $66\frac{2}{3}\%$. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been grouped by topic, but there are roughly equally many mainly motivated by each of the four areas identified in the syllabus (linear algebra; real analysis; probability; discrete mathematics and operations research/optimization). Nor have the problems been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. The examination will begin at 8:30 AM; lunch and refreshments will be provided. The exam will end just before 5:00 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

Exercise 1. Suppose that A is an m by n matrix, and B is an n by s matrix. We denote by C^T the transpose of a matrix C.
Prove that $(AB)^T = B^T A^T$.

Solution: Matrix AB is m by s, and $(AB)^T$ and $B^T A^T$ are both s by m. Now, by the definition of the transpose and the definition of matrix multiplication,

$$\begin{aligned} ((AB)^T)_{ij} &= (AB)_{ji} \\ &= \sum_{k=1}^{n} (A)_{jk}(B)_{ki} \\ &= \sum_{k=1}^{n} (AT)_{kj}(B^T)_{ik} \\ &= \sum_{k=1}^{n} (B^T)_{ik}(AT)_{kj} \\ &= (B^T A^T)_{ij}. \end{aligned}$$

Thus, $(AB)^T = B^T A^T$ as claimed.

Exercise 2. Let A be an $n \times n$ matrix. Show that the rank of A equals 1 if and only if A is of the form ab^T for some choice of nonzero column n-vectors a and b.

Solution: If A has rank 1, then the span of its columns is a 1-dimensional subspace of $\mathbb{R}^n$. Let a nonzero column vector a span this subspace. Then we can write the i-th column of A as $b_i a$ for some choice constants b_i. It cannot be the case that all b_i are zero since this would imply that A is the zero matrix. If b is the column vector with i-th entry being b_i then b is nonzero and we have ab^T.

Conversely, if $A = ab^T$ with both a and b nonzero, then all of the columns of A are multiples of a, hence the rank of A, i.e. the dimension of the column space of A can only be zero or 1. Since some b_i is nonzero, we conclude that the i-th column of A is nonzero so the rank is 1.

Exercise 3. Let $A = \{a_1, a_2, \ldots, a_m\}$ be distinct "demand points" in $\mathbb{R}^n$, and let $W = \{w_1, w_2, \ldots, w_m\}$ be a corresponding set of positive numerical "weights". The Weber Plant Location Problem with these data, $P(A, W)$, calls for determining a point x in $\mathbb{R}^n$ with a minimum sum of weighted Euclidean distances from the demand points, i.e., calls for

$$\text{Minimize } f(x) := w_1|x - a_1| + w_2|x - a_2| + \cdots + w_m|x - a_m|.$$

Show that $P(A, W)$ is a convex program, and that it has at least one solution.

Solution: We first show that the function f is both continuous and convex. That's true because these properties of functions are preserved both under addition and under multiplication by a positive constant. Thus it suffices to use the properties of the Euclidean norm [Syllabus IIIB(5,6)] to show that functions of the form $|x-a|$ have the properties. Applying the Triangle Inequality to $\{x, y, a\}$, gives the continuity-exhibiting

$$\big||x-a| - |y-a|\big| \leq |x-y|.$$

And for all $t \in [0,1]$ we have

$$\begin{aligned} |((1-t)x + ty) - a| &= |(1-t)(x-a) + t(y-a)| \\ &\leq |(1-t)(x-a)| + |t(y-a)| \\ &= |1-t||x-a| + |t||y-a| \\ &= (1-t)|x-a| + t|y-a|, \end{aligned}$$

exhibiting convexity.

The problem is a convex program, since it involves minimizing a convex function over the convex set $\mathbb{R}^n$. To assure existence of an optimal solution, it suffices to show that the minimization of the continuous function f can be confined to a closed bounded subset of $\mathbb{R}^n$.

To obtain such a subset K, choose any point $p \in \mathbb{R}^n$, and define

$$K := \{x \in \mathbb{R}^n : f(x) \leq f(p)\}.$$

Then the minimization can be confined to K since any point not in K is "worse" than member p of K. The subset is closed as a "lower level set" of the continuous function f, and is bounded because

$$f(x) \to \infty \text{ as } |x| \to \infty,$$

this "coercive" property holding because $f(x) \geq w_1|x - a_1|$ and because, by the Triangle Inequality applied to the origin and (x, a_1),

$$|x - a_1| \to \infty \text{ as } |x| \to \infty.$$

Exercise 4. Let X_1 and X_2 have the joint probability density function

$$\begin{aligned} f(x_1, x_2) &= 2 \text{ for } 0 < x_1 < x_2 < 1, \\ &= 0 \text{ elsewhere.} \end{aligned}$$

Let $x_2 \in (0,1)$. Find the conditional mean of X_1, given $X_2 = x_2$.

Solution: The marginal probability density functions are, respectively,

$$\begin{aligned} f_1(x_1) &= \int_{x_1}^{1} 2dx_2 = 2(1-x_1) \text{ for } 0 < x_1 < 1, \\ &= 0 \text{ elsewhere,} \end{aligned}$$

and

$$\begin{aligned} f_2(x_2) &= \int_0^{x_2} 2dx_1 = 2x_2 \quad \text{for} \quad 0 < x_2 < 1, \\ &= 0 \quad \text{elsewhere.} \end{aligned}$$

The conditional probability density function of X_1, given $X_2 = x_2$, is

$$\begin{aligned} f_{1|2}(x_1|x_2) &= 1/x_2 \quad \text{for} \quad 0 < x_1 < x_2 < 1, \\ &= 0 \quad \text{elsewhere.} \end{aligned}$$

Hence the conditional mean of X_1, given $X_2 = x_2$, is

$$\begin{aligned} E[X_1|x_2] &= \int_{-\infty}^{\infty} x_1 f_{1|2}(x_1|x_2)dx_1 \\ &= \int_0^{x_2} x_1(1/x_2)dx_1 \\ &= x_2/2 \quad \text{for} \quad 0 < x_2 < 1. \end{aligned}$$

Exercise 5. Determine whether each of the following two integrals converges or diverges:

$$\int_0^1 \int_0^1 (x^2 + y^2)^{-1/3} dxdy \quad \textit{and} \quad \int_0^1 \int_0^1 (x^2 + y^2)^{-4/3} dxdy$$

Solution: The first integral converges:

$$\begin{aligned} \int_0^1 \int_0^1 (x^2 + y^2)^{-1/3} dxdy &\leq \int\int_{\{x^2+y^2 \leq 2\}} (x^2 + y^2)^{-1/3} dxdy \\ &= \int_0^{2\pi} \int_0^{\sqrt{2}} (r^2)^{-1/3} r dr d\theta \\ &= 2\pi \int_0^{\sqrt{2}} r^{1/3} dr < \infty \end{aligned}$$

The second integral diverges:

$$\begin{aligned} \int_0^1 \int_0^1 (x^2 + y^2)^{-4/3} dxdy &= \frac{1}{4} \int_{-1}^1 \int_{-1}^1 (x^2 + y^2)^{-4/3} dxdy \\ &\geq \frac{1}{4} \int\int_{\{x^2+y^2 \leq 1\}} (x^2 + y^2)^{-4/3} dxdy \\ &= \frac{\pi}{2} \int_0^1 r^{-5/3} dr = \infty \end{aligned}$$

Exercise 6. Let X and Y be independent standard normal random variables. Find the joint distribution of the pair (U, V) where $U = X$ and $V = \frac{Y}{X}$. Use this to show that V has the Cauchy density

$$f(v) = \big(\pi(1+v^2)\big)^{-1}, \quad -\infty < v < \infty.$$

Solution: Define $\phi(x, y) = (x, \frac{y}{x})$. Then the inverse mapping is $\phi^{-1}(u, v) = (u, uv)$ and the magnitude of the Jacobian of ϕ^{-1} is $|u|$. Hence,

$$\begin{aligned} f_{U,V}(u, v) &= f_{X,Y}(\phi^{-1}(u, v))|u| \\ &= \frac{1}{2\pi} e^{-u^2/2} e^{-u^2 v^2/2} \end{aligned}$$

for $-\infty < u, v < \infty$. The density of V is therefore

$$\begin{aligned} f_V(v) &= \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-u^2/2} e^{-u^2 v^2/2} du \\ &= 2\int_0^{\infty} \frac{u}{2\pi} e^{-u^2(1+v^2)/2} du \\ &= \frac{1}{\pi(1+v^2)}, \quad -\infty < v < \infty. \end{aligned}$$

Exercise 7. Let $f(n)$ denote the number of ways of dividing a strip of n unit squares into single squares and dominoes, with $f(0) = 1$ by convention. (A "domino" is a strip consisting of two adjacent squares.) For example, $f(4) = 5$, since

$$4 = 1+1+1+1 = 1+1+2 = 1+2+1 = 2+1+1 = 2+2$$

gives all the partitions of 4 into 1's and 2's. Prove the general relation

$$f(0) + f(1) + \cdots + f(n) = f(n+2) - 1.$$

Solution: We will equate the results from two approaches to counting the number of ways of dividing a strip of $n+2$ squares into squares and dominoes, using at least one domino.

The first approach is to delete, from the set whose size is $f(n+2)$ by definition, the one member corresponding to the division into $n+2$ squares.

For the second approach, consider the set $D(k)$ of those "divisions" whose last domino covers squares $k+1$ and $k+2$, where $0 \leq k \leq n$. There are $f(k)$ of them, since they can cover squares 1 through k in any of $f(k)$ ways, but must cover squares $k+1$ and $k+2$ by a domino, and squares $k+3$ to $n+2$ by single squares. We are "sizing" the union over k of these disjoint sets $D(k)$, and that gives us the LHS of the desired identity.

Exercise 8. A collection of lines $\mathcal{L}$ in $\mathbb{R}^2$ is said to be in *general position* if
(i) each pair of lines in $\mathcal{L}$ intersect in a single point, and
(ii) the intersection of any subcollection of 3 lines in $\mathcal{L}$ is empty.
Any collection of lines $\mathcal{L}$ in general position breaks up $\mathbb{R}^2$ into a certain number of *pieces.* (Formally, these pieces are the connected open sets whose union is $\mathbb{R}^2 \setminus \bigcup_{i=1}^{n} L_i$.) Let $F(\mathcal{L})$ denote the *number* of these pieces. So, for example, if $\mathcal{L}$ consists of only one line, then $F(\mathcal{L}) = 2$.

(a) Show that $F(\mathcal{L})$ depends only on the number of lines in $\mathcal{L}$ so that we can define G_m to be the common value of $F(\mathcal{L})$ for all $\mathcal{L}$ in general position with $|\mathcal{L}| = m$. Write down a recursive formula that the G_m satisfy.

Hint: When a new line is added to the collection $\mathcal{L}$, it must intersect each line already in $\mathcal{L}$ exactly once.

(b) Solve the recursion in (a) to give a closed-form expression for G_m.

Solution: (a) Clearly $G_1 = 2$. Inductively, let $\mathcal{L} = \{L_1, \ldots, L_m\}$ and assume we have shown that $F(\mathcal{L}) = G_m$ independent of the particular lines. Now add another line L_{m+1} to the collection. This line intersects each of the sets $L_i, i = 1, \ldots, m$ at exactly one point by (i) and none of these points coincide by (ii), so that the complement of this set of intersection points in L_{m+1} is a union of $m+1$ disjoint open intervals. Each of these intervals corresponds to a region defined by $\mathcal{L}$ that is broken into 2 pieces, so adding L_{m+1} has the effect of creating $m+1$ new pieces. We conclude that

$$G_{m+1} = G_m + (m+1).$$

(b)

$$G_2 = G_1 + 2$$
$$G_3 = G_2 + 3 = G_1 + 2 + 3$$
$$G_4 = G_3 + 4 = G_1 + 2 + 3 + 4$$

so we see that

$$G_m = G_1 + \sum_{i=2}^{m} i = G_1 - 1 + \sum_{i=1}^{m} i = 1 + m(m+1)/2.$$

Exercise 9. Let G be a finite connected graph. Prove G is a tree if and only if the average degree of a vertex in G is less than 2.

Solution:

(⇒) Suppose G is a tree with n vertices. Then G has $n-1$ edges and so the sum of the degrees of vertices in G is $2(n-1)$. Therefore the average degree of a vertex is $(2n-2)/n = 2-2/n < 2$.

(⇐) Suppose G is a connected graph on n vertices in which the average degree of a vertex is less than 2. Therefore the sum of the degrees of the vertices in G is less than $2n$, and so the number of edges in G is less than n. Since a connected graph on n vertices must have at least $n-1$ edges, it follows that G has exactly $n-1$ edges. Since a connected graph with n vertices and $n-1$ edges must be a tree, G is a tree.

Exercise 10.
Let X be a real $a \times b$ matrix where $a < b$. Find, with proof, $\det(X^T X)$.

Solution: Even if the student has no idea how to do this problem, it should be "obvious" that the answer is zero. Otherwise, if the answer were some other number p, then multiplying X by some scalar would multiply p by some power of that scalar, giving a different result.

But I digress as the above is not a proof. The following will do:

The columns of $X^T X$ are linear combinations of the columns of X^T. Since X^T has a columns, there are at most a linearly independent columns in $X^T X$. Since $X^T X$ is a $b \times b$-matrix where $b > a$, the columns of $X^T X$ are linearly dependent. Therefore $\det(X^T X)$ is zero.

Exercise 11. For the 3×3 matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

calculate the exponential $e^{t\mathbf{A}}$ for real parameter t.

Solution: Direct calculation gives $\mathbf{A}^2 = -\mathbf{E}$, where $\mathbf{E}$ is the rank-2 projection

$$\mathbf{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore, $\mathbf{A}^3 = -\mathbf{A}$, $\mathbf{A}^4 = \mathbf{E}$, etc. Substituting into $e^{t\mathbf{A}} = \sum_{k=0} \frac{t^k \mathbf{A}^k}{k!}$ gives

$$e^{t\mathbf{A}} = \begin{pmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{pmatrix}.$$

Exercise 12. A Lévy stable random variable X with parameter $1/2$ and width $b > 0$ has density

$$f_X(x) = \sqrt{\frac{b}{2\pi x^3}} \exp\left(-\frac{b}{2x}\right)$$

for $x > 0$ and zero otherwise. This variable has the moment-generating function

$$M_X(t) = \exp(-\sqrt{-2bt}), \quad t < 0.$$

Determine the probability law of $S_n = (X_1 + \cdots + X_n)/n^2$, where X_k, $k = 1, ..., n$, are i.i.d. Lévy stable variables as above.

Solution: The generating function of S_n is given by

$$[M_X(t/n^2)]^n = [e^{-\sqrt{-2bt/n^2}}]^n = e^{-\sqrt{-2bt}} = M_X(t).$$

Hence, for each positive integer n, the sum S_n is also a Lévy random variable with parameter $1/2$ and width b. The Law of Large Numbers is not applicable here because the random variable X does not satisfy the condition $E|X| < \infty$ of the LLN, due to the asymptotics $f_X(x) \sim \sqrt{b/2\pi x^3}$ as $x \to \infty$.

Exercise 13.

Let A be an $n \times n$ matrix such that $a_{ik}a_{jk} = a_{kk}a_{ij}$ for all i, j, and k.

(a) Prove that A is symmetric.

(b) Show that if λ is an eigenvalue of A, then $\lambda = 0$ or $\lambda = \operatorname{tr} A$, with $\operatorname{tr}(A) = \sum_{k=1}^{n} a_{kk}$. (Hint: Prove that $A^2 = (\operatorname{tr} A)A$.)

Solution:

(a) First observe that $a_{kk}a_{ij} = a_{ik}a_{jk}$ is invariant under transposition of i and j. If there exists k such that $a_{kk} \neq 0$, then the result follows. If not, then $(a_{ij})^2 = a_{ij}a_{ij} = a_{jj}a_{ii} = 0$, so $a_{ij} = 0$; similarly, $a_{ji} = 0$, so $a_{ij} = a_{ji}$ in this case, too.

(b) Let $B := A^2$. Then, using part (a) at the second equality,

$$b_{ij} = \sum_{k=1}^{n} a_{ik}a_{kj} = \sum_{k=1}^{n} a_{ik}a_{jk} = \sum_{k=1}^{n} a_{kk}a_{ij} = (\operatorname{tr} A)a_{ij},$$

as desired.

If x is an eigenvector of A corresponding to eigenvalue λ, then

$$\lambda^2 x = A^2 x = (\operatorname{tr} A)Ax = (\operatorname{tr} A)\lambda x,$$

from which the desired result is clear.

Exercise 14. The amount of time that a certain type of component functions before failing is a random variable X with probability density function

$$f(x) = 3x^2 I_{\{0<x<1\}}.$$

(a) Calculate the expectation EX.

(b) Once the component fails, it is immediately replaced by another one of the same type. If we let X_i denote the lifetime of the ith component to be put in use, then $S_n = \sum_{i=1}^n X_i$ represents the epoch of the nth failure. The long-term rate at which failures occur, call it R, is defined by

$$R := \lim_{n\to\infty} \frac{n}{S_n}.$$

Assuming that the random variables X_i, $i \geq 1$, are independent, show that R is well defined (with probability one) and determine the distribution of the random variable R.

Solution:

(a) We have $EX = \int x f(x)\,dx = 3\int_0^1 x^3\,dx = 3/4$.

(b) By the strong law of large numbers, with probability one the limit defining R exists and equals $1/EX = 4/3$. Thus the distribution of R is unit mass at $4/3$.

Exercise 15. The joint density function of random variables X and Y is given by

$$f(x,y) = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right)$$

for $0 < x < 1$ and $0 < y < 2$.
(a) Compute the marginal density function of X.
(b) Find $P[X > Y]$.

Solution:
(a) $f_X(x) = \frac{6}{7}\int_0^2 \left(x^2 + \frac{xy}{2}\right) dy = \frac{6}{7}(2x^2 + x)$.

(b) $P[X > Y] = \frac{6}{7}\int_0^1\int_0^x \left(x^2 + \frac{xy}{2}\right) dydx = \frac{5}{16}$.

Exercise 16. When a current I (measured in amperes) flows through a resistance R (measured in ohms), the power generated is given by $W = I^2R$ (measured in watts). Suppose that I and R are independent random variables with densities

$$f_I(x) = 6x(1-x)I\{0 \leq x \leq 1\}$$

and

$$f_R(y) = 2yI\{0 \leq y \leq 1\}.$$

Determine the distribution function of W.

Solution: For $0 < w < 1$,

$$P[I^2R \leq w] = \int\int 6x(1-x)2ydydx$$

where the integration is over the region $x^2y \leq w$, $0 \leq x \leq 1$, $0 \leq y \leq 1$.

Graph the region to help determine limits of integration, and continue:

$$\int_0^{\sqrt{w}}\int_0^1 12x(1-x)ydydx + \int_{\sqrt{w}}^1\int_0^{w/x^2} 12x(1-x)ydydx$$

$$\int_0^{\sqrt{w}} 6x(1-x)dx + \int_{\sqrt{w}}^1 \frac{6(1-x)w^2}{x^3}dx$$

$$= 3w - 2w^{3/2} + 3w^2 - 6w^{3/2} + 3w$$

$$= 6w - 8w^{3/2} + 3w^2.$$

Exercise 17. Show that the set $\{\ x \in \mathbb{R}^3 : x_3 \geq \sqrt{x_1^2 + x_2^2}\ \}$ is a closed convex cone.

Solution:

The function $f(x) = x_3 - \sqrt{x_1^2 + x_2^2}$ being continuous, the set $\Gamma = \{x : f(x) \geq 0\}$ is closed: if $x(n)$ is a sequence in Γ which converges to $x \in \mathbb{R}^3$, then $f(x(n))$ converges to $f(x)$ by continuity, and, since $f(x(n)) \geq 0$ for all n, this is also true for $f(x)$.

To show that it is a convex cone, we must prove that, if $a, b > 0$, and $x, y \in \Gamma$, then $ax+by \in \Gamma$. Denote $z = ax+by$ and let $x' = (x_1, x_2)$, $y' = (y_1, y_2)$ and $z' = (z_1, z_2)$. We have, using well-known properties of the Euclidean norm:

$$\begin{aligned} f(z) &= z_3 - \|z'\| = ax_3 + by_3 - \|ax' + by'\| \\ &\geq ax_3 + by_3 - a\|x'\| - b\|y'\| = af(x) + bf(y) \geq 0 \end{aligned}$$

which proves that $z \in \Gamma$.

One can also prove directly (without using the properties of the Euclidean norm) that $x_3 \geq \sqrt{x_1^2 + x_2^2}$ and $y_3 \geq \sqrt{y_1^2 + y_2^2}$ imply

$$ax_3 + by_3 \geq \sqrt{(ax_1 + by_1)^2 + (ax_2 + by_2)^2}.$$

Since the hypotheses imply $ax_3 + by_3 \geq a\sqrt{x_1^2 + x_2^2}. + b\sqrt{y_1^2 + y_2^2}$, it suffices to show that

$$a\sqrt{x_1^2 + x_2^2}. + b\sqrt{y_1^2 + y_2^2} \geq \sqrt{(ax_1 + by_1)^2 + (ax_2 + by_2)^2}$$

Letting $\lambda = b/a$ and dividing everything by a, this is equivalent to $f(\lambda) \geq 0$ with

$$f(\lambda) = \sqrt{x_1^2 + x_2^2}. + \lambda\sqrt{y_1^2 + y_2^2} - \sqrt{(x_1 + \lambda y_1)^2 + (x_2 + \lambda y_2)^2}.$$

This can be proved by squaring twice the inequalities or by studying the variations of f. Let's detail the second method: we have $f(0) = 0$ and

$$f'(\lambda) = \sqrt{y_1^2 + y_2^2} - \frac{y_1(x_1 + \lambda y_1) + y_2(x_2 + \lambda y_2)}{\sqrt{(x_1 + \lambda y_1)^2 + (x_2 + \lambda y_2)^2}}.$$

The Schwartz inequality implies that

$$y_1(x_1 + \lambda y_1) + y_2(x_2 + \lambda y_2) \leq \sqrt{y_1^2 + y_2^2}\sqrt{(x_1 + \lambda y_1)^2 + (x_2 + \lambda y_2)^2}$$

so that $f'(\lambda) \geq 0$: f is increasing, $f(0) = 0$, thus f is always positive.

Exercise 18. It is known that $\omega = (-1, 0)$ is a solution to the dual of the following linear program. Find the value of α and a solution of the problem:

$$\begin{array}{lrcl} \min & \alpha x_2 + 5x_4 & & \\ \text{s.t.} & x_1 - 2x_2 - x_4 & = & -4 \\ & x_2 - x_3 + x_4 & = & -1 \\ & x_i & \geq & 0 \quad (i = 1, \cdots, 4) \end{array}$$

Solution: The dual problem is

$$\begin{array}{lrcl} \max & -4\omega_1 - \omega_2 & & \\ \text{s.t.} & -2\omega_1 + \omega_2 & \leq & \alpha \\ & -\omega_2 & \leq & 0 \\ & -\omega_1 + \omega_2 & \leq & 5 \end{array}$$

It follows from the complementary slackness principle that $x_1 = 0$ and $x_4 = 0$, because 1st and 3rd dual constraints hold at $\omega = (-1, 0)$ strictly. Using $x_1 = x_4 = 0$ and primal feasibility, we get the system for x_3 and x_3 as follows:

$$\begin{cases} -2x_2 = -4 \\ x_2 - x_3 = -1 \end{cases}$$

This gives $x_2 = 2$ and $x_3 = 3$. To find α, use the complementary slackness principle to get $-2\omega_1 + \omega_2 = \alpha$ because $x_2 > 0$. Therefore $\alpha = 2$.

Exercise 19.

Define, for $(x, y) \in \mathbb{R}^2$, $(x, y) \neq (0, 0)$:

$$f(x, y) = \frac{2 \sin x \sin y}{x^2 + y^2}$$

a) Show that f is bounded and decide if it can be extended to a continuous function on $\mathbb{R}^2$.

b) Let C be the unit disc: $C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Compute

$$\int_C f(x, y) dx dy.$$

Solution: First notice that $|\sin x| \leq |x|$ for all x: this can be considered as known, and retrieved by studying the function $x - \sin x$ for $x > 0$ which is increasing and vanishes at 0. This implies

$$|f(x, y)| \leq |2xy|/(x^2 + y^2) \leq 1$$

since $x^2 + y^2 - 2|xy| = (|x| - |y|)^2 \geq 0$.

The function f does not have a continuous extension: we have $f(0, y) = 0$ and, $f(y, y) \sim 1$ at $y = 0$, which shows that f does not have a limit at $(0, 0)$.

Because f is bounded, the integral is well defined as the limit of the integral on $C_\epsilon = C \setminus \epsilon C$ when $\epsilon \to 0$. The computation is trivial: by symmetry the integral is 0.

Exercise 20.

Let f be a continuous function defined on $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$, such that, for all $X, Y \in \mathbb{R}^2$,

$$f(X, Y, Z) = f(AX + T, AY + T, AZ + T)$$

for all $T \in \mathbb{R}^2$ and all 2 by 2 matrices A with determinant equal to 1 or -1.

Show that there exists a function $g : \mathbb{R}^+ \to \mathbb{R}$ such that

$$f(X, Y, Z) = g(A_{XYZ})$$

where $A_{XYZ} = |\det[Y - X, Z - X]|/2$ is the area of the triangle with vertices X, Y, Z.

(Hint: Use suitably chosen A and T to reduce X, Y, Z into some canonical position.)

Solution:

First consider the case in which $Y - X$ and Z_X are linearly independent. Consider the matrix $A = [Y - X, Z - X]$. If e_1, e_2 is the canonical basis of $\mathbb{R}^2$, we have, by construction $A^{-1}(Y - X) = e_1$ and $A^{-1}(Z - X) = e_2$ and $|\det(A)| = 2A_{XYZ}$. Let $B = \sqrt{|\det(A)|}A^{-1}$ and $T = -B^{-1}X$: we have

$$f(X, Y, Z) = f(BX + T, BY + T, BZ + T) = f(0, 2\sqrt{A_{XYZ}}e_1, 2\sqrt{A_{XYZ}}e_2)$$

and therefore only depends on A_{XYZ}. This now can be extended by continuity to all possible X, Y, Z.