

Department of Applied Mathematics and Statistics  
The Johns Hopkins University

INTRODUCTORY EXAMINATION

Friday, September 3, 2004

**Instructions: Read carefully!**

1. This **closed-book** examination consists of 20 problems (sorry, no choices), each worth 5 points. The passing grade has been set at 75%. Partial credit will be given as appropriate; each part of a problem will be given the same weight unless otherwise specified. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
2. You have been provided with a syllabus indicating the scope of the exam. However, our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.
3. The problems have not been grouped by topic, but there are roughly equally many mainly motivated by each of the four areas identified in the syllabus (linear algebra; real analysis; probability; discrete mathematics and operations research/optimization). Nor have the problems been arranged systematically by difficulty, although you may find the first three easier than most of the others. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM-NUMBER on each sheet.
5. The examination will begin at 8:30 AM; lunch and refreshments will be provided. To avoid “time pressure” and its fear, students may if desired work up to just before 5:00 PM, but the exam has been constructed with the belief and hope that much less time than that will generally be needed.
6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily. **No calculators of any sort are needed or permitted.**

**GOOD LUCK!**

1. Prove that the intersection  $\mathcal{O}_1 \cap \mathcal{O}_2$  of two open sets  $\mathcal{O}_1$  and  $\mathcal{O}_2$  in  $\mathbb{R}^n$  is open.

2. Suppose  $n$  persons arrive at a party, each bearing a gift. The gifts are collected and distributed at random to the  $n$  guests. What is the average number of persons who receive the same gift they brought to the party?

3. Let  $u = [u_1, u_2, u_3]^T$  and  $v = [v_1, v_2, v_3]^T$  be vectors in  $\mathbb{R}^3$ .

(a) Show that the cross product  $u \times v$  is orthogonal to  $u$ .

(b) Show that *Lagrange's identity*

$$\|u \times v\|^2 = \|u\|^2\|v\|^2 - (u \cdot v)^2$$

holds.

4. Let  $x, y \in \mathbb{R}^n$ . Prove that

$$\left| \|x\|^2 - \|y\|^2 \right| \leq \left[ \sum_j (x_j + y_j)^2 \cdot \sum_j (x_j - y_j)^2 \right]^{1/2}.$$

The sums are over all  $j$  from 1 through  $n$ .

5. Prove that if  $E_1, E_2, E_3, \dots, E_n$  are independent events, then

$$\mathbb{P}(\cup_{i=1}^n E_i) = 1 - \prod_{i=1}^n [1 - \mathbb{P}(E_i)].$$

6. Suppose that the coefficients of the power series  $\sum_n a_n x^n$  are integers, infinitely many of which are distinct from zero. Prove that the radius of convergence is at most 1.

7. A trucker has to transport an unstable cargo, in rough terrain, from an origin  $O$  to a destination  $D$  over a network of one-way road-links. Associated with each link  $L$  is a calculable probability  $p(L)$  that the cargo will explode while being transported over  $L$ . Consider the problem of finding a route, from  $O$  to  $D$ , that maximizes the probability that the cargo will complete the trip without exploding.

- (a) **(4 points)** Show how, under an appropriate assumption of independence, this can be transformed into a standard shortest-path problem solvable by the usual methods for such problems.
- (b) **(1 point)** Under what (physical) circumstances might the assumption of independence be unreasonable?

8. Let  $(a_n : n = 1, 2, \dots)$  be a sequence of real numbers defined recursively:  $a_1 = \sqrt{2}$  and  $a_{n+1} = \sqrt{2 + a_n}$  for  $n \geq 1$ . Show that

$$\lim_{n \rightarrow \infty} a_n = A$$

exists and find  $A$ .

9. Suppose that a cancer diagnostic test is 95 percent accurate both on those that do and those that do not have the disease. If 0.4 percent of the population have cancer, compute the probability that a tested person chosen randomly from the population has cancer, given that his or her test result indicates so.

10. Let  $D \subseteq \mathbb{R}^p$ , and suppose that  $(f_n)$  is a sequence of continuous  $\mathbb{R}^q$ -valued functions that converges uniformly on  $D$  to  $f$ . Suppose also that  $(x_n)$  is a sequence of elements of  $D$  that converges to  $x$  in  $D$ . Prove that  $f_n(x_n) \rightarrow f(x)$ .

11. For a nonnegative integer  $k$ , let  $a_k := 2k + 1$  and let  $b_k := k^2$ . Set

$$F(x) := \sum_{k=0}^{\infty} a_k x^k \quad \text{and} \quad G(x) := \sum_{k=0}^{\infty} b_k x^k.$$

Prove that

$$G(x) = \frac{x}{1-x} F(x).$$

[Note: You need not discuss the convergence of the various series encountered.]

**12.** If  $X$  and  $Y$  are real-valued random variables with joint density  $f_{X,Y}$ , show that the density function  $f_Z$  of the product  $Z = XY$  is given by the improper integral

$$f_Z(z) = \int_{-\infty}^{+\infty} f_{X,Y}(x, z/x) |x|^{-1} dx.$$

**13.** Let

$$A = \begin{bmatrix} 1 & \rho \\ \rho & \rho^2 \end{bmatrix},$$

where  $\rho \in \mathbb{R}$ .

- (a) Compute the eigenvalues and eigenvectors of this matrix.
- (b) For what values of  $\rho$  is  $A$  positive definite?
- (c) For what values of  $\rho$  is  $A$  positive semidefinite?

**14.** Given real numbers  $x$  and  $y$ , the so-called  $n$ -dimensional *combinatorial matrix*  $A$  is defined as the  $n \times n$  matrix

$$A := xI + yJ,$$

where  $I$  is the identity matrix and  $J$  is the matrix of all 1's.

- (a) Compute the determinant of  $A$ .
- (b) Show that  $A$  is invertible if and only if  $x \neq 0$  and  $x + ny \neq 0$ , and that

$$A^{-1} = \frac{1}{x(x + ny)} [(x + ny)I - yJ]$$

in that case.

**15.** Let  $n \geq 3$  be an integer and let  $N = \{1, 2, \dots, n\}$ . Form a graph  $G$  in which  $V(G)$  is the set of all 2-element subsets of  $N$  and in which (for distinct vertices  $v$  and  $w$ )  $vw \in E(G)$  provided  $v \cap w$  is nonempty. Prove that  $G$  is Eulerian.

**16.** Suppose that  $f : [0, 1] \rightarrow [0, 1]$  is such that

$$|f(x) - f(y)| \geq |x - y| \text{ for all } x, y \in [0, 1]. \quad (*)$$

- (a) If  $f$  is continuous, show that  $f$  is one-to-one and onto.
- (b) Find all functions  $f : [0, 1] \rightarrow [0, 1]$  satisfying the property (\*).

**17.** A fair six-sided die is continually rolled until the sum of the outcomes of all rolls (strictly) exceeds 300. Is the probability that at least 82 rolls are needed smaller or larger than 75%? Explain your answer. [HINT: You may use without proof the fact that the standard deviation of the outcome of a single roll is  $\sqrt{35/12} \approx 1.7078$ .]

**18.** LUXO is a company that manufactures luxury cars and trucks. It believes its most likely customers are high-income women and men (HIW's and HIM's, respectively). To reach these groups, it's undertaking an advertising campaign involving the purchase of one-minute commercial spots on two types of TV programs: romantic comedies and football games. Each comedy commercial costs \$50K, and is estimated to be seen by 7 million HIW's and 2 million HIM's; the corresponding figures for "football ads" are: \$100K, 2 million HIW's, and 12 million HIM's. The company would like its commercials to reach at least 28 million HIW's and 24 million HIM's. Initial issue: how many spots of each type (comedy or football) to buy?

- (a) Formulate a naive linear program to determine how LUXO should seek to meet its goals at minimum total cost.
- (b) Solve the model by whatever method you like, but with a clear explanation.
- (c) Identify at least three potential pitfalls in using your model as a valid reliable tool for reaching the company's "what to do?" decision.

**19.** The *convex hull* of a set  $X \subseteq \mathbb{R}^n$  is defined as the set of points of the form  $\sum_{i=1}^k \lambda_i x_i$  where, for each  $i$ , we have  $x_i \in X$  and  $\lambda_i \geq 0$ , and where  $\sum_{i=1}^k \lambda_i = 1$ . Denote the convex hull of  $X$  by  $\text{conv}(X)$ . Let  $X \subseteq \mathbb{R}^n$  be a finite set, and let  $C := \text{conv}(X)$ .

- (a) Prove that  $C$  is closed under the formation of convex combinations, that is, if  $x, y \in C$ , then  $\lambda x + (1 - \lambda)y \in C$  for all  $\lambda \in [0, 1]$ .
- (b) Prove that  $C$  is a closed set.

**20.** Use the spectral theorem for positive semidefinite real symmetric matrices (or more generally for real symmetric matrices) to prove that if  $A = [a_{ij}]$  and  $B = [b_{ij}]$  are  $n \times n$  positive semidefinite real symmetric matrices, then so is  $C = [a_{ij}b_{ij}]$ . [HINT: Determine what the spectral theorem gives for the form of the entries  $a_{ij}$ .]