Department of Applied Mathematics and Statistics
The Johns Hopkins University

INTRODUCTORY EXAMINATION–FALL SESSION

August 29, 2007

Instructions: Read carefully!

1. This **closed-book** examination consists of 20 problems (sorry, no choices), each worth 5 points. The passing grade has been set at $66\frac{2}{3}\%$. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been grouped by topic, but there are roughly equally many mainly motivated by each of the four areas identified in the syllabus (linear algebra; real analysis; probability; discrete mathematics and operations research/optimization). Nor have the problems been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. The examination will begin at 8:30 AM; lunch and refreshments will be provided. The exam will end just before 5:00 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Prove that there is no graph with 11 vertices such that both G and $\overline{G}$ are planar.

   Note: $\overline{G}$ denotes the complement of G.

Solution

Suppose, for the sake of contradiction, that there is a graph with 11 vertices such that G and $\overline{G}$ are planar.

Since G and $\overline{G}$ are planar, they have at most $3 \times 11 - 6 = 27$ edges. So, taken together, they have at most 54 edges. However, this does not account for the $\binom{11}{2} = 55$ edges in K_{11}. This is a contradiction.

2. Let

$$A = \begin{bmatrix} 0 & \sqrt{2} & 1 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}$$

   Find the $v \in \mathbb{R}^3$ that maximize the Euclidean norm $\| Av \|$ subject to $\| v \|^2 = 1$.

Solution

We are maximizing $v^T A^T Av$ subject to $v^T v = 1$, and using Lagrange multipliers the solutions satisfy $A^T Av = \lambda v$ for some constant v, that is, any solution must be an eigenvector of $A^T A$. Furthermore, a solution v satisfies $v^T (A^T A)v = \lambda v^T v = \lambda$ so the optimal choice is to take v to be a unit eigenvector corresponding to the largest eigenvalue.

We calculate

$$A^t A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & \sqrt{2} \\ 0 & \sqrt{2} & 3 \end{bmatrix}.$$

The characteristic polynomial of this matrix is $\lambda(\lambda^2 - 5\lambda + 4) = \lambda(\lambda - 1)(\lambda - 4)$, so the largest eigenvalue is $\lambda = 4$. A quick calculation shows that the eigenspace corresponding to this eigenvector is spanned by

$$v = \begin{bmatrix} 0 \\ \sqrt{2} \\ 1 \end{bmatrix},$$

and we obtain the solution by normalizing

$$v = \pm \begin{bmatrix} 0 \\ \sqrt{2}/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}.$$

3. A function $f : \mathbb{R}^p \to \mathbb{R}$ is said to be *lower semi-continuous* at c if

$$f(c) \le \liminf_{x \to c} f(x).$$

(a) Show that if A is an open subset of $\mathbb{R}^p$, then its indicator function is everywhere lower semi-continuous.

(b) Is the converse true? If so, prove it; if not, give an explicit counterexample.

Solution

(a) Let f denote the indicator function of A. If $c \notin A$, then the desired inequality is trivially satisfied. If $c \in A$, then $x \in A$, and consequently $f(x) = 1$, for all x sufficiently close to c; hence

$$f(c) = 1 = \lim_{x \to c} f(x) = \liminf_{x \to c} f(x).$$

(b) The converse is true. Indeed, if $c \in A$, then, by the definition of lower semi-continuity, $f(x) = 1$, i.e., $x \in A$, for all x sufficiently close to c.

4. 1) Let A, B, C be 3 events from a probability space Ω and P a probability measure on Ω. Prove that

$$P((A \cap B) \cup (A \cap C) \cup (B \cap C)) = P(A) + P(B) + P(C) - P(A \cup B \cup C) - P(A \cap B \cap C)$$

2) For a given illness, a patient is symptomatic when at least one out of 3 symptoms is present. The patient is at stage 2 of the illness when two of the symptoms are present, and stage 3 if all 3 are present. In the symptomatic population, 30% have symptom 1, 45% have symptom 2, and 60% have symptom 3. Moreover, 5% of this population have all three symptoms together.

Compute the proportion of patients at stage 2 within the symptomatic population.

Solution

$$\begin{aligned} P((A \cap B) \cup (A \cap C) \cup (B \cap C)) &= P(A \cap B) + P(A \cap C) + P(B \cap C) \\ &\quad - P((A \cap B) \cap (A \cap C)) - P((A \cap B) \cap (B \cap C)) - P((A \cap C) \cap (B \cap C)) \\ &\quad + P((A \cap B) \cap (A \cap C) \cap (B \cap C)) \\ &= P(A \cap B) + P(A \cap C) + P(B \cap C) - 2P(A \cap B \cap C) \end{aligned}$$

Now

$$P(A \cap B) + P(A \cap C) + P(B \cap C) = P(A) + P(B) + P(C) - P(A \cup B \cup C) + P(A \cap B \cap C)$$

which provides the result.

The previous result must be applied with A: symptom 1, B: symptom 2 and C: symptom 3. It provides the probability that at least two symptoms are present, which is

$$0.30+0.45+0.60-1-0.05=0.3$$

($P(A\cup B\cup C)=1$ since one only considers the symptomatic population.)

The proportion of the population at stage 2 is obtained by substracting the proportion at stage 3, and is 25%

5. Let $\|x\|$ denote the Euclidean norm of a vector x in $\mathbb{R}^n$. Show that if p and q are nonzero vectors in $\mathbb{R}^n$, then
$$\left\|\frac{p}{\|p\|}-\frac{q}{\|q\|}\right\| \leq \frac{\|p-q\|}{\min\{\|p\|,\|q\|\}}.$$

Solution

The inequality is obviously true if $\|p\|=\|q\|$. Without loss of generality, assume $\|p\|<\|q\|$. Then multiplying through by $\|p\|$ gives the equivalence of the above inequality and the inequality
$$\left\|p-\frac{\|p\|}{\|q\|}q\right\| \leq \|p-q\|.$$
Since both sides are nonnegative, squaring gives the further equivalence of the inequality with
$$\|p\|^2+\frac{\|p\|^2}{\|q\|^2}\|q\|^2-2\frac{\|p\|}{\|q\|}\langle p,q\rangle \leq \|p\|^2+\|q\|^2-2\langle p,q\rangle,$$
Hence
$$2\left(1-\frac{\|p\|}{\|q\|}\right)\langle p,q\rangle \leq \left(1-\frac{\|p\|^2}{\|q\|^2}\right)\|q\|^2.$$
Dividing the above by $1-\frac{\|p\|}{\|q\|}>0$, we get another equivalent inequality:

$$2\langle p,q\rangle \leq \left(1+\frac{\|p\|}{\|q\|}\right)\|q\|^2=\|p\|\|q\|+\|q\|^2.$$

The result then follows directly from Cauchy-Schwartz.

6. A primitive model for heat conduction leads to the equation

$$\begin{bmatrix}u_n\\v_n\end{bmatrix}=A^n\begin{bmatrix}u_0\\v_0\end{bmatrix}, n=1,2,\cdots,A=\begin{bmatrix}7/9 & 1/9\\1/9 & 7/9\end{bmatrix},$$

for temperatures $u_n = u(n\Delta t)$ and $v_n = v(n\Delta t)$.
Find the limits c_u and c_v, where $u_n \to c_u$ and $v_n \to c_v$ as $n \to \infty$.
Proceed as follows:
(a) Find the eigenvalues of A.
(b) Represent A^n as CDC^{-1}, where D is a diagonal matrix.
(c) Use this representation of A^n to find the limits c_u and c_v.

Solution

(a) A has eigenvalues $8/9$ and $2/3$.
(b) $A = CBC^{-1}$ where $B = diag(8/9, 2/3)$. Therefore,

$$A^2 = (CBC^{-1})(CBC^{-1}) = CB^2C^{-1},$$

$$A^3 = (CBC^{-1})(CB^2C^{-1}) = CB^3C^{-1},$$

and, in general,

$$A^n = CB^nC^{-1}.$$

Since B is a diagonal matrix, $B^n = diag((8/9)^n, (2/3)^n)$.
(c) Since $(8/9)^n \to 0$ and $(2/3)^n \to 0$ when $n \to \infty$, the elements of B^n approach 0 as limit when $n \to \infty$. Since C and C^{-1} are fixed, so do the elements of $A^n = CB^nC^{-1}$ approach 0 as limit when $n \to \infty$. Therefore, $u_n \to 0$ and $v_n \to 0$ as $n \to \infty$. That is, $c_u = c_v = 0$.

7. A standard Cauchy random variable has density function

$$f(x) = \frac{1}{\pi(1+x^2)}, \qquad -\infty < x < \infty.$$

If X is a standard Cauchy random variable, find the density function of the random variable $\frac{1}{X}$.

Solution

For $t > 0$,

$$P\left[\frac{1}{X} \le t\right] = P\left[X \ge \frac{1}{t}\right] = \int_{\frac{1}{t}}^{\infty} \frac{1}{\pi(1+x^2)} dx.$$

By differentiating with respect to the lower limit of integration, using the chain rule,

$$\frac{d}{dt}\left(\int_{\frac{1}{t}}^{\infty} \frac{1}{\pi(1+x^2)} dx\right)$$

$$= \frac{1}{\pi\left(1+\left(\frac{1}{t}\right)^2\right)}\left(-\frac{d}{dt}\left(\frac{1}{t}\right)\right)$$

$$= \frac{1}{\pi\left(1+\left(\frac{1}{t}\right)^2\right)}\left(\frac{1}{t^2}\right)$$

$$= \frac{1}{\pi(t^2+1)}$$

For $t < 0$, since the density of X is symmetric about 0, so is the density of $\frac{1}{X}$.

8. (a) Find the two singular values of the 2×2 matrix

$$A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix},$$

for any $a \in \mathbb{R}$.

(b) Find the product of the singular values *without using the explicit values in (a).*

Solution

(a) The eigenvalues of

$$A^*A = \begin{pmatrix} 1+a^2 & a \\ a & 1 \end{pmatrix},$$

satisfy the characteristic equation

$$\lambda^2 - (2+a^2)\lambda + 1 = 0,$$

with the solutions

$$\sigma_\pm^2 = \lambda_\pm = 1 + \frac{1}{2}a^2 \pm a\sqrt{1+\frac{1}{4}a^2},$$

where $\sigma_\pm$ are the singular values.

(b) Since

$$\sigma_+^2\sigma_-^2 = \det(A^*A) = [\det(A)]^2 = 1$$

and since the singular values are, by definition, non-negative,

$$\sigma_+\sigma_- = 1.$$

9. Let A be an $m \times n$ real matrix, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. Consider the following linear program:

$$\begin{array}{ll} \min & c^T x \\ s.t. & Ax = b \end{array}$$

Show that the linear program has an optimal solution if and only if $b \in Range(A)$ and $c \in Range(A^T)$.

Solution

If $b \in Range(A)$ then the feasible region is not empty. If $c \in Range(A^T)$ then there exists a vector z such that $c = A^T z$, then for any feasible point x we have $c^T x = z^T Ax = z^T b$. Therefore, the objective function is constant in the feasible region and hence an optimal solution exists.

Conversely, if an optimal solution $\hat{x}$ exists then $b = A\hat{x}$ and $b \in Range(A)$. For any $d \in Null(A)$, $\hat{x} + \alpha d$ is feasible for any real α. It follows from $c^T(\hat{x} + \alpha d) \geq c^T\hat{x}$ that $c^T d = 0$. Therefore, c is in the orthogonal complement of $Null(A)$ and hence $z \in range(A^T)$.

10. Let A be a full-rank $n \times k$ matrix. Prove that the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ by $T(w) = A(A^T A)^{-1} A^T w$ is the orthogonal projection onto the column space of A.

Solution

Let V denote the column space of A. For any $w \in \mathbb{R}^n$, $T(w)$ is of the form Au where $u = (A^T A)^{-1} A^T w$, which is a linear combination of the columns of A. Thus, T maps $\mathbb{R}^n$ into V. We need to check that $w - T(w)$ is orthogonal to every element of V for every $w \in \mathbb{R}^n$. If u is any element of V we can write $u = Av$ for some $v \in \mathbb{R}^k$ and we have

$$(w - T(w))^T u = (w - T(w))^T (Av) = (w - A(A^T A)^{-1} A^T w)^T (Av) = (w^T - w^T A(A^T A)^{-1} A^T)(Av)$$
$$= w^T Av - w^T A(A^T A)^{-1} A^T Av = w^T Av - w^T Av = 0.$$

11. Approximate the smallest number of people needed so that the probability that at least one of them has the same birthday as you is greater than one-half. (You may neglect February 29).

Solution

Assume that the individuals have birthdays which are uniformly distributed over the 365 days of the year (neglecting February 29), independently.

If there are n other people, the number of birthdays identical to your birthday among them has a Binomial(n,1/365) distribution. Since the success probability is small, if n is large, this distribution can be approximated by a Poisson(n/365) distribution. If we denote such a random variable by X, we want to find the smallest n such that $P[X \geq 1] \geq 1/2$ or equivalently $P[X = 0] < 1/2$. Thus, we solve

$$e^{-n/365} < 1/2$$

to obtain $n > 365 \log 2 \approx 253$.

12. It is known that $w = (-1, 0)$ is an optimal solution to the dual of the following linear program, find the value of α and an optimal solution of the probelm:

$$\begin{array}{lrcl} \min & \alpha x_2 \quad + 5x_4 & & \\ \text{s.t.} & x_1 - 2x_2 \quad - x_4 & = & -4 \\ & x_2 - x_3 + x_4 & = & -1 \\ & x_i & \geq & 0 \quad (i = 1, \cdots, 4) \end{array}$$

Solution

We first find the dual problem:

$$\begin{array}{lrcl} \max & -4w_1 - w_2 & & \\ \text{s.t.} & w_1 & \leq & 0 \\ & -2w_1 + w_2 & \leq & \alpha \\ & -w_2 & \leq & 0 \\ & -w_1 + w_2 & \leq & 5. \end{array}$$

Notice that the first and the fourth constraints are satisfied strictly at $w(-1, 0)$, therefore it follows from the complementary slackness principle that $x_1 = x_4 = 0$. Then use $x_1 = x_4 = 0$ and two equations in the primal constraints, we can solve for x and get $x_2 = 2$ and $x_3 = 3$. To find α we use the fact that the two optimal objective function values are equal; that is $4 = 2\alpha$. Hence, $\alpha = 2$.

13. For a nonnegative integer n, define

$$f(n) = \sum_{k=0}^{n} \binom{n-k}{k}.$$

For example,

$$f(5) = \binom{5}{0} + \binom{4}{1} + \binom{3}{2} + \binom{2}{3} + \binom{1}{4} + \binom{0}{5} = 1 + 4 + 3 + 0 + 0 + 0 = 8.$$

The sequence $f(0), f(1), f(2), \ldots$ begins $1, 1, 2, 3, 5, 8, 13, \ldots$, and so we suspect that $f(n)$ is the nth Fibonacci number. Prove that this is so.

Solution

The proof is by (strong) induction on n. The basis cases $n = 0$ and $n = 1$ are trivial. We now must show that $f(n+2) = f(n+1) + f(n)$ for all n. Suppose we have shown that $f(n)$ is the nth Fibonacci number and that $f(n+1)$ is the $(n+1)$st Fibonacci number. Observe that

$$\begin{array}{rl} f(n) = & \qquad\qquad \binom{n}{0} + \binom{n-1}{1} + \cdots + \binom{1}{n-1} + \binom{0}{n} \\ +f(n+1) = & \binom{n+1}{0} + \binom{n}{1} + \binom{n-1}{2} + \cdots + \binom{1}{n} + \binom{0}{n+1} \\ \hline \end{array}$$

$$\begin{aligned} f(n)+f(n+1) &= \binom{n+1}{0} + \left[\binom{n}{0} + \binom{n}{1}\right] + \left[\binom{n-1}{1} + \binom{n-1}{2}\right] + \cdots \\ &= \binom{n+2}{0} + \binom{n+1}{1} + \binom{n}{2} + \cdots \\ &= f(n+2) \end{aligned}$$

where we have used Pascal's identity $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$. Therefore $f(n+2)$ is the $(n+2)$nd Fibonacci number, and so the result follows by induction.

14. Suppose that u, v are elements of an inner-product space over the complex field. If $\|\cdot\|$ is the norm defined by the inner product, are the following values possible?

$$\|u+v\| = 1, \ \|u+iv\| = 2, \ \|u-v\| = 1, \ \|u-iv\| = 2.$$

If so, find an example. If not, prove that the values are impossible.

Solution

By the polarization identity, the inner product

$$\langle u, v\rangle = \frac{1}{4}(\|u+v\|^2 - \|u-v\|^2) + \frac{1}{4}i(\|u+iv\|^2 - \|u-iv\|^2).$$

Hence, with the above values, $\langle u, v\rangle = 0$ and u, v must be orthogonal. But, in that case, all the four given norms should equal the same value $\sqrt{\|u\|^2 + \|v\|^2}$. Thus, the stated values are impossible.

15. Let $\{x_n, n = 1, 2, \ldots\}$ and $\{y_n, n = 1, 2, \ldots\}$ be two positive sequences of real numbers for which

$$\frac{x_{n+1}}{x_n} \leq \frac{y_{n+1}}{y_n}, \ n = 1, 2, \ldots$$

Show that if $\sum_n y_n$ is convergent then so is $\sum_n x_n$.

Solution

We have

$$\frac{x_2}{x_1} \leq \frac{y_2}{y_1}, \ \frac{x_3}{x_2} \leq \frac{y_3}{y_2}, \ldots, \frac{x_n}{x_{n-1}} \leq \frac{y_n}{y_{n-1}}.$$

Hence,

$$x_n \leq C y_n, \ n = 1, 2, \ldots$$

where $C = \frac{x_1}{y_1} > 0$, from which convergence follows easily.

16. let $n = 2m$ where m is a positive integer, and for $p \in [0,1]$ let X_p be a Binomial(n, p) random variable. For $x \in \{0, \cdots, n\}$, let

$$\lambda(x) = \frac{P[X_{x/n} = x]}{P[X_{1/2} = x]}.$$

(a) Show that $\lambda(\cdot)$ is symmetric about m.
(b) Show that $\lambda(\cdot)$ is decreasing on $\{0, \cdots, m\}$.

Solution

$$\lambda(x) = \frac{\binom{n}{x}(x/n)^x(1-x/n)^{n-x}}{\binom{n}{x}(1/2)^x(1-1/2)^{n-x}} = 2^n(x/n)^x(1-x/n)^{n-x} = 2^n(x/n)^x((n-x)/n)^{n-x}.$$

(a)

$$\begin{aligned}
\lambda(m+y) &= 2^n((m+y)/n)^{m+y}(1-(m+y)/n)^{n-(m+y)} \\
&= 2^n(1-(m-y)/n)^{n-(m-y)}((m-y)/n)^{m-y} \\
&= \lambda(m-y)
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{\partial}{\partial x}\log\lambda(x) &= \frac{\partial}{\partial x}[n\log 2 + x\log(x/n) + (n-x)\log((n-x)/n)] \\
&= ((x)(n/x)(1/n) + \log(x/n)) + ((n-x)(n/(n-x))(-1/n) - \log((n-x)/n)) \\
&= 1 + \log(x/n) - 1 - \log((n-x)/n) \\
&= \log(x/n) - \log((n-x)/n) \\
&= \log((x/n)/((n-x)/n)) \\
&= \log(x/(n-x)).
\end{aligned}$$

Therefore, $x/(n-x) \leq 1 \iff x \leq n/2$ with strict inequality $\iff x < n/2$, and thus $\frac{\partial}{\partial x}\log\lambda(x) \leq 0 \iff x \leq n/2$ with strict inequality $\iff x < n/2$. It follows that $\lambda(\cdot)$ is decreasing on $\{0, \cdots, m = n/2\}$ as desired.

17. Must a (pointwise) nondecreasing limit of nonnegative Riemann integrable functions on $[0,1]$ be Riemann integrable? If so, then prove this. If not, then give an explicit counterexample.

Solution

The answer is no. For a counterexample, let $\{r_1, r_2, \dots\}$ be an enumeration of the rational numbers in $I := [0,1]$. For $n \geq 1$, define f_n to be the indicator of $\{r_1, \dots, r_n\}$. Then f_n is Riemann integrable on I (with vanishing integral), and the sequence (f_n) converges monotonically up to the indicator f of the rationals. It is well known (and easy to prove) that f is *not* Riemann integrable.

18. The number of births per day in a small town's hospital follows the following distribution

# births	0	1	2	3	4
probability	.25	.45	.14	.11	.05

Assume that a baby has a probability 1/2 to be a girl. What is the most likely number of births in a day if it is known that exactly 2 girls are born ?

Solution

Let Y be the variable equal to the number of girls born during a day. We must compute $P(X = x|Y = 2)$:

$$P(X = x|Y = 2) \quad = \quad P(X = x \text{ and } Y = 2)/P(Y = 2)$$

Since we need only the most likely x, the computation of $P(Y = 2)$ is not needed. We have $P(X = x \text{ and } Y = 2) = P(Y = 2|X = x)P(X = x)$. This is (for $x > 1$)

$$P(X = x \text{ and } Y = 2) = \binom{x}{2} \times 0.5^2 \times 0.5^{x-2} P(X = x) = \binom{x}{2} \times 0.5^x P(X = x)$$

For $x = 2$, this is $0.14/4$. For $x = 3 : 3(0.11)/8$ and for $x = 4 : 6(0.05)/16$. The maximum is at $x = 3$.

19. Let $\{a_1, \ldots, a_M\}$ be a set of positive real numbers. Show that

$$\lim_{n\to\infty} \frac{1}{n} \ln \sum_{m=1}^{M} a_m^n = \max_{1\leq m\leq M} \ln a_m.$$

Solution

Let $\bar{a} = \max_{1\leq m\leq M} a_m$. Then:

$$\bar{a}^n \leq \sum_{m=1}^{M} a_m^n \leq M\bar{a}^n$$

and consequently

$$\ln \bar{a} \leq \frac{1}{n} \ln \sum_{m=1}^{M} a_m^n \leq \frac{\ln M}{n} + \ln \bar{a}.$$

It follows that the limit exists and equals $\ln \bar{a} = \max_{1\leq m\leq M} \ln a_m$.

20. Let A be an $n \times n$ real orthogonal matrix, that is $A^{-1} = A^T$.

    (a) Show that A, regarded as a linear transformation, preserves the (Euclidean) length of every vector.

    (b) Show that the only possible eigenvalues of A corresponding to real eigenvectors are 1 and -1.

Solution

(a) Since orthogonality implies $A^T A = I$, for any vector $x \in \mathbf{R}^n$ we have

$$\|Ax\|^2 = (Ax)^T(Ax) = x^T A^T Ax = x^T x = \|x\|^2,$$

implying the desired result $\|Ax\| = \|x\|$.

(b) We will apply part (a) to a real eigenvector x for some eigenvector L of A. It follows from

$$Ax = Lx,$$

with at least one $x_i \neq 0$, that L is real. Applying part (a) and using the above equation, we have

$$\|x\| = \|Ax\| = \|Lx\| = \|L\|\|x\|$$

so that $\|L\| = 1$, giving (since L is real) the desired result.

[NOTE to graders: For full credit in part (b), I feel it is necessary to show that L is real.]