On Vertex, Edge, and Vertex-Edge Random Graphs (Extended Abstract)

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Abstract

We consider three classes of random graphs: edge random graphs, vertex random graphs, and vertex-edge random graphs. Edge random graphs are Erdős-Rényi random graphs [9, 10], vertex random graphs are generalizations of geometric random graphs [21], and vertex-edge random graphs generalize both. The names of these three types of random graphs describe where the randomness in the models lies: in the edges, in the vertices, or in both. We show that vertex-edge random graphs, ostensibly the most general of the three models, can be approximated arbitrarily closely by vertex random graphs, but that the two categories are distinct.

1 Introduction

The classic random graphs are those of Erdős and Rényi [9, 10]. In their model, each edge is chosen independently of every other. The randomness inhabits the edges; vertices simply serve as placeholders to which random edges attach.

Since the introduction of Erdős-Rényi random graphs, many other models of random graphs have been developed. For example, *random geometric graphs* are formed by randomly assigning points in a Euclidean space to vertices and then adding edges deterministically between vertices when the distance between their assigned points is below a fixed threshold; see [21] for an overview. For these random graphs, the randomness inhabits the vertices and the edges reflect relations between the randomly chosen structures assigned to them.

Finally, there is a class of random graphs in which randomness is imbued both upon the vertices and upon the edges. For example, in latent position models of social networks, we imagine each vertex as assigned to a random position in a metric "social" space. Then, given the positions, vertices whose points are near each other are more likely to be adjacent. See, for example, [2, 13, 17, 18, 20]. Such random graphs are, roughly speaking, a hybrid of Erdős-Rényi and geometric graphs.

We call these three categories, respectively, edge random, vertex random, and vertex-edge random graphs. From their formal definitions in Section 2, it follows immediately that vertex random and edge random graphs are instances of the more generous vertex-edge random graph models. But is the vertex-edge random graph category strictly more encompassing? We observe in Section 3 that a vertex-edge random graph can be approximated arbitrarily closely by a vertex random graph. Is it possible these two categories are, in fact, the same? The answer is no, and this is presented in Section 4. Our discussion closes in Section 5 with some open problems. All nontrivial proofs excluded from this extended abstract can be found in the corresponding full-length paper [3].

Nowadays, in most papers on random graphs, for each value of n a distribution is placed on the collection of n-vertex graphs and asymptotics as $n \to \infty$ are studied. We emphasize that in this extended abstract, by contrast, the focus is on what kinds of distributions arise in certain ways for a single arbitrary but fixed value of n.

2 Random Graphs

For a positive integer n, let $[n] = \{1, 2, \ldots, n\}$ and let $\mathscr{G}_n$ denote the set of all simple graphs $G = (V, E)$ with vertex set $V = [n]$. (A simple graph is an undirected graph with no loops and no parallel edges.) We often abbreviate the edge (unordered pair) $\{i, j\}$ as ij or write $i \sim j$ and say that i and j are adjacent.

When we make use of probability spaces, we omit discussion of measurability when it is safe to do so. For example, when the sample space is finite it goes without saying that the corresponding σ-field is the total σ-field, that is, that all subsets of the sample space are taken to be measurable.

DEFINITION 2.1. (RANDOM GRAPH) A *random graph* is a probability space of the form $\mathbf{G} = (\mathscr{G}_n, P)$ where n is a positive integer and P is a probability measure defined on $\mathscr{G}_n$.

In actuality, we *should* define a random graph as a graph-valued random variable, that is, as a measurable mapping from a probability space into $\mathscr{G}_n$. However, the distribution of such a random object is a probability measure on $\mathscr{G}_n$ and is all that is of interest in this extended abstract, so the abuse of terminology in Definition 2.1 serves our purposes.

Example 2.1. (ERDŐS-RÉNYI RANDOM GRAPHS) A simple random graph is the Erdős-Rényi random graph in the case $p = \frac{1}{2}$.

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This is the random graph $\mathbf{G} = (\mathscr{G}_n, P)$ where

$$P(G) := 2^{-\binom{n}{2}}, \quad G \in \mathscr{G}_n.$$

[Here and throughout we abbreviate $P(\{G\})$ as $P(G)$; this will cause no confusion.] More generally, an Erdős-Rényi random graph is a random graph $\mathbf{G}(n,p) = (\mathscr{G}_n, P)$ where $p \in [0,1]$ and

$$P(G) := p^{|E(G)|}(1-p)^{\binom{n}{2}-|E(G)|}, \quad G \in \mathscr{G}_n.$$

This means that the $\binom{n}{2}$ potential edges appear independently of each other, each with probability p.

This random graph model was first introduced by Gilbert [12]. Erdős and Rényi [9, 10], who started the systematic study of random graphs, actually considered a closely related model with a fixed number of edges. However, it is now common to call both models Erdős-Rényi random graphs.

Example 2.2. (SINGLE COIN-FLIP RANDOM GRAPHS) Another simple family of random graphs is one we call the *single coin-flip* family. Here $\mathbf{G} = (\mathscr{G}_n, P)$ where $p \in [0,1]$ and

$$P(G) := \begin{cases} p & \text{if } G = K_n, \\ 1-p & \text{if } G = \overline{K_n}, \\ 0 & \text{otherwise.} \end{cases}$$

As in the preceding example, each edge appears with probability p; but now all edges appear or none do.

In the successive subsections we specify our definitions of *edge*, *vertex*, and *vertex-edge* random graphs.

2.1 Edge random graph In this extended abstract, by an edge random graph (abbreviated ERG in the sequel) we simply mean a classical Erdős-Rényi random graph.

DEFINITION 2.2. (EDGE RANDOM GRAPH) An *edge random graph* is an Erdős-Rényi random graph $\mathbf{G}(n,p)$.

We shall also make use of the following generalization that allows variability in the edge-probabilities.

DEFINITION 2.3. (GENERALIZED EDGE RANDOM GRAPH) A generalized edge random graph (GERG) is a random graph for which the events that individual vertex-pairs are joined by edges are mutually independent but do not necessarily have the same probability. Thus to each pair $\{i,j\}$ of distinct vertices we associate a probability $\mathbf{p}(i,j)$ and include the edge ij with probability $\mathbf{p}(i,j)$; edge random graphs are the special case where $\mathbf{p}$ is constant.

Formally, a GERG can be described in the following manner. Let n be a positive integer and let $\mathbf{p} : [n] \times [n] \to [0,1]$ be a symmetric function. The *generalized edge random graph* $\mathbf{G}(n,\mathbf{p})$ is the probability space $(\mathscr{G}_n, P)$ with

$$P(G) := \prod_{\substack{i<j \\ ij \in E(G)}} \mathbf{p}(i,j) \times \prod_{\substack{i<j \\ ij \notin E(G)}} [1-\mathbf{p}(i,j)].$$

We call the graphs in these two definitions (generalized) *edge* random graphs because all of the randomness inhabits the (potential) edges. The inclusion of ERGs in GERGs is strict, as easily constructed examples show.

GERGs have appeared previously in the literature, e.g. in [1]; see also the next example and Definition 2.7 below.

As discussed in the next example, GERGs have appeared previously in the literature.

Example 2.3. (STOCHASTIC BLOCKMODEL RANDOM GRAPHS) A stochastic blockmodel random graph is a GERG in which the vertex set is partitioned into blocks $B_1, B_2, \ldots, B_b$ and the probability that vertices i and j are adjacent depends only on the blocks in which i and j reside.

A simple example is a random bipartite graph defined by partitioning the vertex set into B_1 and B_2 and taking $\mathbf{p}(i,j) = 0$ if $i,j \in B_1$ or $i,j \in B_2$, while $\mathbf{p}(i,j) = p$ (for some given p) if $i \in B_1$ and $j \in B_2$ or vice versa.

The concept of blockmodel is interesting and useful when b remains fixed and $n \to \infty$. Asymptotics of blockmodel random graphs have been considered, for example, by Söderberg [25]. (He also considers the version where the partitioning is random, constructed by independent random choices of a type in $\{1,...,b\}$ for each vertex; see Example 2.8.)

Recall, however, that in this extended abstract we hold n fixed and note that in fact every GERG can be represented as a blockmodel by taking each block to be a singleton.

A salient feature of Example 2.3 is that vertex labels matter. Intuitively, we may expect that if all isomorphic graphs are treated "the same" by a GERG, then it is an ERG. We proceed to formalize this correct intuition, omitting the simple proof of Proposition 2.1.

DEFINITION 2.4. (ISOMORPHISM INVARIANCE) Let $\mathbf{G} = (\mathscr{G}_n, P)$ be a random graph. We say that $\mathbf{G}$ is *isomorphism-invariant* if for all $G, H \in \mathscr{G}_n$ we have $P(G) = P(H)$ whenever G and H are isomorphic.

PROPOSITION 2.1. *Let* $\mathbf{G}$ *be an isomorphism-invariant generalized edge random graph. Then* $\mathbf{G} = \mathbf{G}(n,p)$ *for some* n,p. *That is,* $\mathbf{G}$ *is an edge random graph.*

2.2 Vertex random graph The concept of a vertex random graph (abbreviated VRG) is motivated by the idea of a random intersection graph. One imagines a universe $\mathscr{S}$ of geometric objects. A random $\mathscr{S}$-graph $G \in \mathscr{G}_n$ is created by choosing n members of $\mathscr{S}$ independently at random[1], say $S_1, \ldots, S_n$, and then declaring distinct vertices i and j to be adjacent if and only if $S_i \cap S_j \neq \emptyset$. For example, when $\mathscr{S}$ is the set of real intervals, one obtains a random interval graph [6, 15, 22, 23]; see Example 2.5 for more. In [11, 16, 24] one takes $\mathscr{S}$ to consist of discrete (finite) sets. Random chordal graphs can be defined by selecting random subtrees of a tree [19].

[1] Of course, some probability distribution must be associated with $\mathscr{S}$.

Notice that for these random graphs, all the randomness lies in the structures attached to the vertices; once these random structures have been assigned to the vertices, the edges are *determined*. In Definition 2.6 we generalize the idea of a random intersection graph to other vertex-based representations of graphs; see [29].

DEFINITION 2.5. ($(\mathbf{x},\phi)$-GRAPH) Let n be a positive integer, $\mathscr{X}$ a set, $\mathbf{x} = (x_1,\dots,x_n)$ a function from $[n]$ into $\mathscr{X}$, and $\phi : \mathscr{X} \times \mathscr{X} \to \{0,1\}$ a symmetric function. Then the $(\mathbf{x},\phi)$-*graph*, denoted $\mathbf{G}(\mathbf{x},\phi)$, is defined to be the graph with vertex set $[n]$ such that for all $i,j \in [n]$ with $i \neq j$ we have

$$ij \in E \quad \text{if and only if} \quad \phi(x_i,x_j) = 1.$$

Of course, every graph $G = (V,E)$ with $V = [n]$ is an $(\mathbf{x},\phi)$-graph for some choice of $\mathscr{X}$, $\mathbf{x}$, and ϕ; one need only take $\mathbf{x}$ to be the identity function on $\mathscr{X} := [n]$ and define

$$\phi(i,j) := \mathbf{1}(ij \in E) = \begin{cases} 1 & \text{if } ij \in E \\ 0 & \text{otherwise.} \end{cases}$$

It is also clear that this representation of G as an $(\mathbf{x},\phi)$-graph is far from unique. The notion of $(\mathbf{x},\phi)$-graph becomes more interesting when one or more of $\mathscr{X}$, $\mathbf{x}$, and ϕ are specified.

Example 2.4. (INTERVAL GRAPHS) Take $\mathscr{X}$ to be the set of all real intervals and define

$$\phi(J,J') := \begin{cases} 1 & \text{if } J \cap J' \neq \emptyset \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

In this case, an $(\mathbf{x},\phi)$-graph is exactly an interval graph.

DEFINITION 2.6. (VERTEX RANDOM GRAPH) To construct a vertex random graph (abbreviated VRG), we imbue $\mathscr{X}$ with a probability measure μ and sample n elements of $\mathscr{X}$ independently at random to get $\mathbf{x}$, and then we build the $(\mathbf{x},\phi)$-graph.

Formally, let n be a positive integer, $(\mathscr{X},\mu)$ a probability space, and $\phi : \mathscr{X} \times \mathscr{X} \to \{0,1\}$ a symmetric function. The *vertex random graph* $\mathbf{G}(n,\mathscr{X},\mu,\phi)$ is the random graph $(\mathscr{G}_n,P)$ with

$$P(G) := \int \mathbf{1}\{\mathbf{G}(\mathbf{x},\phi) = G\}\,\mu(d\mathbf{x}), \quad G \in \mathscr{G}_n,$$

where $\mu(d\mathbf{x})$ is shorthand for the product integrator $\mu^n(d\mathbf{x}) = \mu(dx_1)\dots\mu(dx_n)$ on $\mathscr{X}^n$.

Note that $\mathbf{G}(\cdot,\phi)$ is a graph-valued random variable defined on $\mathscr{X}^n$. The probability assigned by the vertex random graph to $G \in \mathscr{G}_n$ is simply the probability that this random variable takes the value G.

Example 2.5. (RANDOM INTERVAL GRAPHS) Let $\mathscr{X}$ be the set of real intervals as in Example 2.4, let ϕ be as in (2.1), and let μ be a probability measure on $\mathscr{X}$. This yields a VRG that is a random interval graph.

Example 2.6. (RANDOM THRESHOLD GRAPHS) Let $\mathscr{X} = [0,1]$, let μ be Lebesgue measure, and let ϕ be the indicator of a given up-set U in the usual (coordinatewise) partial order $\preceq$ on $\mathscr{X} \times \mathscr{X}$ (i.e., a set U such that $y \in U$ and $y \leq z$ implies $z \in U$). This yields a VRG that is a random threshold graph; see [7].

Example 2.7. (RANDOM GEOMETRIC GRAPHS) Random geometric graphs are studied extensively in [21]. Such random graphs are created by choosing n i.i.d. (independent and identically distributed) points from some probability distribution on $\mathbf{R}^k$. Then, two vertices are joined by an edge exactly when they lie within a certain distance, t, of each other.

Expressed in our notation, we let $(\mathscr{X},d)$ be a metric space equipped with a probability measure μ and let $t > 0$ (a threshold). For points $x,y \in \mathscr{X}$ define

$$\phi(x,y) := \mathbf{1}\{d(x,y) \leq t\}.$$

That is, two vertices are adjacent exactly when the distance between their corresponding randomly chosen points is sufficiently small.

Because the n vertices in a vertex random graph are drawn i.i.d. from $(\mathscr{X},\mu)$, it is easy to see that the random graph is isomorphism-invariant.

PROPOSITION 2.2. *Every vertex random graph is isomorphism-invariant.*

2.3 Vertex-edge random graphs A generalization both of vertex random graphs and of edge random graphs are the *vertex-edge* random graphs (abbreviated VERGs) of Definition 2.8. First we generalize Definition 2.5 to allow edge probabilities other than 0 and 1.

DEFINITION 2.7. (RANDOM $(\mathbf{x},\phi)$-GRAPH) Given a positive integer $n \geq 1$, a set $\mathscr{X}$, and a function $\phi : \mathscr{X} \times \mathscr{X} \to [0,1]$, we assign to each $i \in [n]$ a deterministically chosen object $x_i \in \mathscr{X}$. Then, for each pair $\{i,j\}$ of vertices, independently of all other pairs, the edge ij is included in the random $(\mathbf{x},\phi)$-graph with probability $\phi(x_i,x_j)$.

Formally, let $\mathbf{x} = (x_1,\dots,x_n)$ be a given function from $[n]$ into $\mathscr{X}$. Then the *random* $(\mathbf{x},\phi)$-*graph*, denoted $\mathbf{G}(\mathbf{x},\phi)$, is defined to be the random graph $(\mathscr{G}_n,P_{\mathbf{x}})$ for which the probability of $G \in \mathscr{G}_n$ is given by

$$P_{\mathbf{x}}(G) := \prod_{i<j,\ i\sim j} \phi(x_i,x_j) \times \prod_{i<j,\ i\not\sim j} [1-\phi(x_i,x_j)].$$

Notice that $\mathbf{G}(\mathbf{x},\phi)$ is simply the generalized edge random graph $\mathbf{G}(n,\mathbf{p})$ where $\mathbf{p}(i,j) := \phi(x_i,x_j)$ (recall Definition 2.3).

DEFINITION 2.8. (VERTEX-EDGE RANDOM GRAPH) Let n be a positive integer, $(\mathscr{X},\mu)$ a probability space, and $\phi : \mathscr{X} \times \mathscr{X} \to [0,1]$ a symmetric function. In words, a vertex-edge random graph is generated like this: First a list of random elements is drawn i.i.d., with distribution μ, from $\mathscr{X}$; call the list $\mathbf{X} = (X_1,\dots,X_n)$.

Then, conditionally given $\mathbf{X}$, independently for each pair of distinct vertices i and j we include the edge ij with probability $\phi(X_i, X_j)$.

Formally, the *vertex-edge random graph* $\mathbf{G}(n, \mathscr{X}, \mu, \phi)$ is the random graph $(\mathscr{G}_n, P)$ with

$$P(G) := \int P_{\mathbf{x}}(G)\, \mu(d\mathbf{x})$$

where the integration notation is as in Definition 2.6 and $P_{\mathbf{x}}$ is the probability measure for the random $(\mathbf{x}, \phi)$-graph $\mathbf{G}(\mathbf{x}, \phi)$ of Definition 2.7.

Note that a VRG is the special case of a VERG with ϕ taking values in $\{0, 1\}$.

It can be shown [14] that every VERG can be constructed with the standard choice $\mathscr{X} = [0,1]$ and $\mu =$ Lebesgue measure. However, other choices are often convenient in specific situations.

We note in passing that one could generalize the notions of VRG and VERG in the same way that edge random graphs (ERGs) were generalized in Definition 2.3, by allowing different functions ϕ_{ij} for different vertex pairs $\{i, j\}$. But while the notion of generalized ERG was relevant to the definition of a VERG (recall the sentence preceding Definition 2.8), we neither study nor employ generalized VRGs and VERGs in this extended abstract.

Asymptotic properties (as $n \to \infty$) of random $(\mathbf{x}, \phi)$-graphs and VERGs have been studied by several authors: see, e.g., [4] and the references therein. VERGs are also important in the theory of *graph limits*; see for example [5, 8, 18].

Example 2.8. (FINITE-TYPE VERG) In the special case when $\mathscr{X}$ is finite, $\mathscr{X} = \{1, \ldots, b\}$ say, we thus randomly and independently choose a type in $\{1, \ldots, b\}$ for each vertex, with a given distribution μ; we can regard this as a random partition of the vertex set into blocks $B_1, \ldots, B_b$ (possibly empty, and with sizes governed by a multinomial distribution). A VERG with $\mathscr{X}$ finite can thus be regarded as a stochastic blockmodel graph with multinomial random blocks; cf. Example 2.3. Such finite-type VERGs have been considered by Söderberg [25, 26, 27, 28].

Example 2.9. (RANDOM DOT PRODUCT GRAPHS) In [17, 20] random graphs are generated by the following two-step process. First, n vectors (representing n vertices) $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are chosen i.i.d. according to some probability distribution on $\mathbf{R}^k$. With this choice in place, distinct vertices i and j are made adjacent with probability $\mathbf{v}_i \cdot \mathbf{v}_j$. All pairs are considered (conditionally) independently. Care is taken so that the distribution on $\mathbf{R}^k$ satisfies

$$P\big(\mathbf{v}_i \cdot \mathbf{v}_j \notin [0,1]\big) = 0.$$

Random dot product graphs are vertex-edge random graphs with $\mathscr{X} = \mathbf{R}^k$ and $\phi(\mathbf{v}, \mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$.

As with vertex random graphs, all vertices are treated "the same" in the construction of a vertex-edge random graph.

PROPOSITION 2.3. *Every vertex-edge random graph is isomorphism-invariant.*

Note that we use the notation $\mathbf{G}(n, \mathscr{X}, \mu, \phi)$ for both VRGs and VERGs. This is entirely justified because ϕ takes values in in $\{0, 1\}$ for VRGs and in $[0, 1]$ for VERGs. If perchance the ϕ function for a VERG takes only the values 0 and 1, then the two notions coincide. Hence we have part (b) of the following proposition; part (a) is equally obvious.

PROPOSITION 2.4.

(a) Every edge random graph is a vertex-edge random graph.

(b) Every vertex random graph is a vertex-edge random graph.

However, not all generalized edge random graphs are vertex-edge random graphs, as simple counterexamples show.

We now ask whether the converses to the statements in Proposition 2.4 are true. The converse to Proposition 2.4(a) is false. Indeed, It is easy to find examples of VERGs that aren't ERGs:

Example 2.10. We present one small class of examples of VERGs that are even VRGs, but not ERGs. Consider *random interval graphs* [6, 15, 22] $\mathbf{G}(n, \mathscr{X}, \mu, \phi)$ with $n \geq 3$, $\mathscr{X}$ and ϕ as in Example 2.4, and (for $i \in [n]$) the random interval J_i corresponding to vertex i constructed as $[X_i, Y_i]$ or $[Y_i, X_i]$, whichever is nonempty, where $X_1, Y_1, \ldots, X_n, Y_n$ are i.i.d. uniform$[0,1]$ random variables. From an elementary calculation, independent of n, one finds that the events $\{1 \sim 2\}$ and $\{1 \sim 3\}$ are not independent.

The main result of this extended abstract (Theorem 4.1; see also the stronger Theorem 4.2) is that the converse to Proposition 2.4(b) is also false. The class of vertex random graphs does not contain the class of vertex-edge random graphs; however, as shown in the next section, every vertex-edge random graph can be approximated arbitrarily closely by a vertex random graph.

An overview of the inclusions of these various categories is presented in Figure 1.

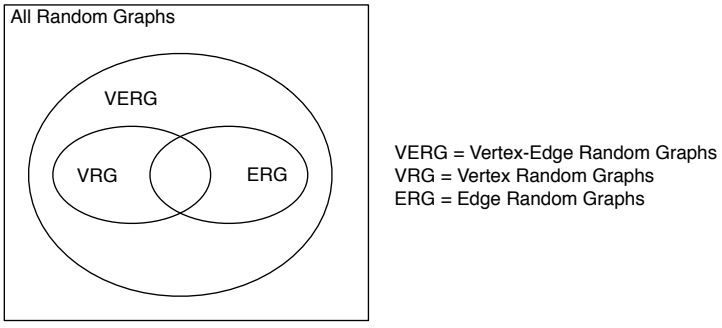

Figure 1: Venn diagram of random graph classes. The results of this extended abstract show that all five regions in the diagram are nonempty.

3 Approximation

The goal of this section is to show that every vertex-edge random graph can be closely approximated by a vertex random graph. Our notion of approximation is based on total variation distance. (This choice is not important. We consider a fixed n, and the space of probability measures on $\mathscr{G}_n$ is a finite-dimensional simplex, and thus compact. Hence any continuous metric on the probability measures on $\mathscr{G}_n$ is equivalent to the total variation distance, and can be used in Theorem 3.1.)

DEFINITION 3.1. (TOTAL VARIATION DISTANCE) Let $\mathbf{G}_1 = (\mathscr{G}_n, P_1)$ and $\mathbf{G}_2 = (\mathscr{G}_n, P_2)$ be random graphs on n vertices. We define the *total variation distance* between $\mathbf{G}_1$ and $\mathbf{G}_2$ to be

$$d_{\mathrm{TV}}(\mathbf{G}_1, \mathbf{G}_2) := \frac{1}{2} \sum_{G \in \mathscr{G}_n} |P_1(G) - P_2(G)| .$$

Total variation distance can be reexpressed in terms of the maximum discrepancy of the probability of events.

PROPOSITION 3.1. *Let $\mathbf{G}_1 = (\mathscr{G}_n, P_1)$ and $\mathbf{G}_2 = (\mathscr{G}_n, P_2)$ be random graphs on n vertices. Then*

$$d_{\mathrm{TV}}(\mathbf{G}_1, \mathbf{G}_2) = \max_{B \subseteq \mathscr{G}_n} |P_1(B) - P_2(B)| .$$

THEOREM 3.1. *Let $\mathbf{G}$ be a vertex-edge random graph and let $\varepsilon > 0$. There exists a vertex random graph $\widehat{\mathbf{G}}$ with $d_{\mathrm{TV}}(\mathbf{G}, \widehat{\mathbf{G}}) < \varepsilon$.*

The proof of Theorem 3.1 is given in the Appendix.

4 Not all vertex-edge random graphs are vertex random graphs

In Section 3 (Theorem 3.1) it was shown that every vertex-edge random graph can be approximated arbitrarily closely by a vertex random graph. This naturally raises the question of whether every vertex-edge random graph *is* a vertex random graph. We originally believed that some suitable "$M = \infty$ modification" of the proof of Theorem 3.1 would provide a positive answer, but in fact the answer is no:

THEOREM 4.1. *Not all vertex-edge random graphs are vertex random graphs.*

This theorem is an immediate corollary of the following much stronger result. We say that an ERG $\mathbf{G}(n,p)$ is *nontrivial* when $p \notin \{0,1\}$.

THEOREM 4.2. *If $n \geq 4$, no nontrivial Erdős-Rényi random graph is a vertex random graph. In fact, an ERG $\mathbf{G}(n,p)$ with $n \geq 4$ is represented as a vertex-edge random graph $\mathbf{G}(n, \mathscr{X}, \mu, \phi)$ if and only if $\phi(x,y) = p$ for μ-almost every x and y.*

The first sentence of Theorem 4.2 follows immediately from the second. The "if" part of the second sentence is trivial (for any value of n), since $\phi(x,y) = p$ clearly gives a representation (which we shall call the *canonical* representation) of an ERG as a VERG. The "only if" part of the second sentence is proved in the full-length paper [3].

Consider an ERG $\mathbf{G}(n,p)$. If $n \geq 4$, Theorem 4.2 shows that $\mathbf{G}(n,p)$ is never a VRG if $p \notin \{0,1\}$. Curiously, however, every $\mathbf{G}(n,p)$ with $n \leq 3$ is a VRG; in fact, the following stronger result is true.

THEOREM 4.3. *Every vertex-edge random graph with $n \leq 3$ is a vertex random graph.*

5 Open problems

Call a VERG $\mathbf{G}(n, \mathscr{X}, \mu, \phi)$ *binary* if $\Pr\{\phi(X_1, X_2) \in \{0,1\}\} = 1$ where X_1 and X_2 are independent draws from μ. Since μ-null sets do not matter, this amounts to saying that ϕ gives a representation of the random graph as a VRG.

In Theorem 4.3 we have seen that every VERG with $n \leq 3$ is a VRG, but what is the situation when $n \geq 4$?

OPEN PROBLEM 5.1. *Is there any VRG with $n \geq 4$ that also has a non-binary VERG representation?*

Theorem 4.2 rules out constant-valued non-binary VERG representations ϕ, and the main goal now is to see what other VERGs we can rule out as VRGs. In the following proposition, X_1 and X_2 (respectively, Y_1 and Y_2) are independent draws from μ (respectively, ν).

PROPOSITION 5.1. *If a VRG $\mathbf{G}(n, \mathscr{Y}, \nu, \psi)$ has a representation as a VERG $\mathbf{G}(n, \mathscr{X}, \mu, \phi)$, then ϕ is binary if and only if $\mathbf{E}\,\psi^2(Y_1, Y_2) = \mathbf{E}\,\phi^2(X_1, X_2)$.*

The expression $\mathbf{E}\,\phi^2(X_1, X_2)$ is the squared Hilbert–Schmidt norm of the integral operator $(Tg)(x) := \int \phi(x,y) g(y)\, \mu(dy) = \mathbf{E}[\phi(x, X) g(X)]$ and equals the sum $\sum_i \lambda_i^2$ of squared eigenvalues. So the proposition has the following corollary.

COROLLARY 5.1. *If a VRG $\mathbf{G}(n, \mathscr{Y}, \nu, \psi)$ has a representation as a VERG $\mathbf{G}(n, \mathscr{X}, \mu, \phi)$, and if the respective multisets of nonzero squared eigenvalues of the integral operators associated with ψ and ϕ are the same, then ϕ is binary.*

OPEN PROBLEM 5.2. *Is there any VERG with $n \geq 4$ having two representations with distinct multisets of nonzero squared eigenvalues?*

By Corollary 5.1, a positive answer to Open Problem 5.1 would imply a positive answer to Open Problem 5.2.

Our next result, Proposition 5.2, goes a step beyond Theorem 4.2. We say that ϕ is of rank r when the corresponding integral operator has exactly r nonzero eigenvalues (counting multiplicities). For ϕ to be of rank at most 1 it is equivalent that there exists $0 \leq g \leq 1$ (μ-a.e.) such that (for μ-almost every x_1 and x_2)

$$\phi(x_1, x_2) = g(x_1) g(x_2). \tag{5.2}$$

PROPOSITION 5.2. *For $n \geq 6$, no non-binary VERG $\mathbf{G}(n, \mathscr{X}, \mu, \phi)$ with ϕ of rank at most 1 is a VRG.*

With the hypothesis of Proposition 5.2 strengthened to $n \geq 8$, we can generalize that proposition substantially as follows.

PROPOSITION 5.3. *For $1 \leq r < \infty$ and $n \geq 4(r+1)$, no non-binary VERG $\mathbf{G}(n, \mathscr{X}, \mu, \phi)$ with ϕ of rank at most r is a VRG.*

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A Appendix: Proof of Theorem 3.1

To prove Theorem 3.1 we use the following simple birthday-problem subadditivity upper bound. Let M be a positive integer.

LEMMA A.1. *Let $\mathbf{A} = (A_1, A_2, \ldots, A_n)$ be a random sequence of integers with each A_i chosen independently and uniformly from $[M]$. Then*

$$P\{\mathbf{A} \text{ has a repetition}\} \leq \frac{n^2}{2M}.$$

Proof of Theorem 3.1. Let $\mathbf{G}$ be a vertex-edge random graph on n vertices and let $\varepsilon > 0$. Let M be a large positive integer. (We postpone our discussion of just how large to take M until needed.)

The vertex-edge random graph $\mathbf{G}$ can be written $\mathbf{G} = \mathbf{G}(n, \mathscr{X}, \mu, \phi)$ for some set $\mathscr{X}$ and mapping $\phi : \mathscr{X} \times \mathscr{X} \to [0,1]$.

We construct a vertex random graph $\widehat{\mathbf{G}} = \mathbf{G}(n, \mathscr{Y}, \nu, \psi)$ as follows. Let $\mathscr{Y} := \mathscr{X} \times [0,1]^M \times [M]$; that is, $\mathscr{Y}$ is the set of ordered triples (x, f, a) where $x \in \mathscr{X}$, $f \in [0,1]^M$, and $a \in [M]$. We endow $\mathscr{Y}$ with the product measure of its factors; that is, we independently pick $x \in \mathscr{X}$ according to μ, a function $f \in [0,1]^{[M]}$ uniformly, and $a \in [M]$ uniformly. We denote this measure by ν.

We denote the components of the vector $f \in [0,1]^M$ by $f(1), \ldots, f(M)$, thus regarding f as a random function from $[M]$ into $[0,1]$. Note that for a random $f \in [0,1]^M$, the components

$f(1),\ldots,f(M)$ are i.i.d. random numbers with a uniform$[0,1]$ distribution.

Next we define ψ. Let $y_1,y_2 \in \mathscr{Y}$ where $y_i = (x_i,f_i,a_i)$ (for $i=1,2$). Let

$$\psi(y_1,y_2) = \begin{cases} 1 & \text{if } a_1 < a_2 \text{ and } \phi(x_1,x_2) \geq f_1(a_2), \\ 1 & \text{if } a_2 < a_1 \text{ and } \phi(x_1,x_2) \geq f_2(a_1), \\ 0 & \text{otherwise.} \end{cases}$$

Note that ψ maps $\mathscr{Y} \times \mathscr{Y}$ into $\{0,1\}$ and is symmetric in its arguments. Therefore $\widehat{\mathbf{G}}$ is a vertex random graph.

We now show that $d_{\mathrm{TV}}(\mathbf{G},\widehat{\mathbf{G}})$ can be made arbitrarily small by taking M sufficiently large.

Let $B \subseteq \mathscr{G}_n$. Recall that

$$P(B) = \int P_{\mathbf{x}}(B)\,\mu(d\mathbf{x}),$$
$$\widehat{P}(B) = \int \mathbf{1}\{\mathbf{G}(\mathbf{y},\psi) \in B\}\,\nu(d\mathbf{y}) = \Pr\{\mathbf{G}(\mathbf{Y},\psi) \in B\},$$

where in the last expression the n random variables comprising $\mathbf{Y} = (Y_1,\ldots,Y_n)$ are independently chosen from $\mathscr{Y}$, each according to the distribution ν.

As each Y_i is of the form (X_i,F_i,A_i) we break up the integral for $\widehat{P}(B)$ based on whether or not the a-values of the Ys are repetition free and apply Lemma A.1:

$$\begin{aligned} \widehat{P}(B) &= \Pr\{\mathbf{G}(\mathbf{Y},\psi) \in B \mid \mathbf{A} \text{ is repetition free}\} \\ &\qquad \times \Pr\{\mathbf{A} \text{ is repetition free}\} \\ &\quad + \Pr\{\mathbf{G}(\mathbf{Y},\psi) \in B \mid \mathbf{A} \text{ is not repetition free}\} \\ &\qquad \times \Pr\{\mathbf{A} \text{ is not repetition free}\} \\ &= \Pr\{\mathbf{G}(\mathbf{Y},\psi) \in B \mid \mathbf{A} \text{ is repetition free}\} + \delta \end{aligned} \tag{6.3}$$

where $|\delta| \leq n^2/(2M)$.

Now, for any repetition-free $\mathbf{a}$, the events $\{i \sim j \text{ in } \mathbf{G}(\mathbf{Y},\psi)\}$ are conditionally independent given $\mathbf{X}$ and given $\mathbf{A} = \mathbf{a}$, with

$$\begin{aligned} &\Pr\{i \sim j \text{ in } \mathbf{G}(\mathbf{Y},\psi) \mid \mathbf{X},\ \mathbf{A} = \mathbf{a}\} \\ &= \begin{cases} \Pr\{\phi(X_i,X_j) \geq F_i(a_j) \mid X_i,X_j\} & \text{if } a_i < a_j \\ \Pr\{\phi(X_i,X_j) \geq F_j(a_i) \mid X_i,X_j\} & \text{if } a_j < a_i \end{cases} \\ &= \phi(X_i,X_j). \end{aligned}$$

Thus, for any repetition-free $\mathbf{a}$,

$$\Pr\{\mathbf{G}(\mathbf{Y},\psi) \in B \mid \mathbf{X},\ \mathbf{A} = \mathbf{a}\}$$

equals

$$\sum_{G \in B} \left(\prod_{i<j,\ ij \in E(G)} \phi(X_i,X_j) \times \prod_{i<j,\ ij \notin E(G)} [1-\phi(X_i,X_j)] \right) = P_{\mathbf{X}}(B).$$

Removing the conditioning on $\mathbf{X}$ and $\mathbf{A}$, (6.3) thus implies

$$\widehat{P}(B) = P(B) + \delta,$$

and so $|P(B) - \widehat{P}(B)| \leq n^2/M$ for all $B \subseteq \mathscr{G}_n$. Equivalently, $d_{\mathrm{TV}}(\mathbf{G},\widehat{\mathbf{G}}) \leq n^2/M$. Thus we need only choose $M > n^2/\varepsilon$.