1. Kraichnan in 1976 considered a model of “vortex blobs” with initial stream function of the form

$$\psi_0(x) = \frac{\omega_0}{k_0^2} e^{-r^2/2D^2} \cos(k_0 \cdot x)$$

in an external strain field corresponding to a large-scale velocity

$$U_1(x,t) = a(t) x_1, \quad U_2(x,t) = -a(t) x_2, \quad a(t) > 0.$$ 

(a) Show that the initial velocity and vorticity of the blob are given by

$$u_0(x) = \frac{k_0^\perp}{k_0} \omega_0 e^{-r^2/2D^2} \sin(k_0 \cdot x), \quad \omega_0(x) = \omega_0 e^{-r^2/2D^2} \cos(k_0 \cdot x)$$

up to corrections of order $O((k_0D)^{-1})$ for $k_0D \gg 1$, and likewise the energy and enstrophy are given to leading order by

$$E_0 = \frac{1}{k_0^2} \Omega_0, \quad \Omega_0 = \frac{\pi \omega_0^2 D^2}{4}. $$

Hint: You may find useful the Bessel-function integral $\int_0^\infty du e^{-u} J_0(2\sqrt{\alpha u}) = e^{-\alpha}$. E.g. see A. Erdélyi, Higher Transcendental Functions, vol. I, formula 6.10.8.

(b) Show that under passive distortion $(\partial_t + U \cdot \nabla)\omega = 0$ by the large-scale field, vorticity evolves into $\omega(x,t) = \omega_0 \exp[-(e^{-2\beta(t)}x_1^2 + e^{2\beta(t)}x_2^2)/2D^2] \cos(k(t) \cdot x)$, an elongated elliptical-shaped packet, and the velocity likewise evolves into

$$u(x,t) = \frac{k_+^\perp(t)}{k_+^2(t)} \omega_0 \exp[-(e^{-2\beta(t)}x_1^2 + e^{2\beta(t)}x_2^2)/2D^2] \sin(k(t) \cdot x),$$

with $k(t) = (e^{-\beta(t)}k_1^{(0)}, e^{\beta(t)}k_2^{(0)})$ and $\beta(t) = \int_0^t a(s) ds$, and thus the energy and enstrophy evolve into

$$E(t) = \frac{1}{k_+^2(t)} \Omega_0, \quad \Omega(t) = \Omega_0.$$ 

(c) Now assume that there is a random ensemble of vortex blobs with an isotropic distribution of wavenumbers $k$ of magnitude $k_0$. Show that $\langle k^2(t) \rangle = \cosh[2\beta(t)]k_0^2$, but that

$$\langle E(t) \rangle = \langle k^{-2}(t) \rangle \Omega_0 = k_0^{-2} \Omega_0 = E_0,$$

so that mean energy of the small-scale blobs is conserved! Why doesn’t this result contradict the “vortex-thinning mechanism” of 2D inverse energy cascade?
2. This problem concerns the quantity for an incompressible 2D velocity field $\mathbf{u}$ defined by

\[
\nabla \cdot \sigma^\ell(\mathbf{u}, \mathbf{u}) = \frac{1}{\ell^2} \int d^2r \left( \partial_i \partial_j^\perp G^\ell(r) \delta u_i(r) \delta u_j(r) \right)
- \frac{1}{\ell^2} \int d^2r \left( \partial_i G^{\ell}(r) \delta u_i(r) \right) \cdot \int d^2r \left( \partial_i \partial_j^\perp G^\ell(r) \delta u_j(r) \right)
- \frac{1}{\ell^2} \int d^2r \left( \partial_j G^{\ell}(r) \delta u_j(r) \right) \cdot \int d^2r \left( \partial_i G^\ell(r) \delta u_i(r) \right)
\]

that appears in the DiPerna-Lions theory applied to 2D Euler. Show that if $\mathbf{u}$ is a smooth field, then the first term on the right converges to $-\epsilon_{jk}(D^2)_{jk}$, the second term to 0, and the third to $\epsilon_{jk}(D^2)_{jk}$, with $D_{ij} = \partial u_i / \partial x_j$. Thus, the net limit is zero. In fact, using Homework #5, Problem 3 (a), show that $\epsilon_{jk}(D^2)_{jk} = 0$.

3. Derive the following typical estimate used in the DiPerna-Lions theory:

\[
\|h(\bar{\omega}^\ell) - h(\omega)\|_1 \leq 2^{(p-1)/p} \max \left\{ M_h, 2^{p-1} C \|\omega\|^p \right\} \cdot \|\bar{\omega}^\ell - \omega\|_p
\]

for $h \in \mathcal{H}_p = \{ h : h \in C^1(\mathbb{R}) \& \ |h'(z)| \leq C|z|^{p-1} \text{ for } |z| \geq R \}$. Here we define as usual $\|f\|_p = \left[ \frac{1}{|T|} \int_T d^d x \ |f(x)|^p \right]^{1/p}$.  

**Hint:** Use the mean-value theorem to write

\[
h(\bar{\omega}^\ell(x)) - h(\omega(x)) = h'(\omega^\ell(x)) \cdot (\bar{\omega}^\ell(x) - \omega(x))
\]

where $\omega^\ell(x) = \theta(x)\bar{\omega}^\ell(x) + (1 - \theta(x))\omega(x)$. Use Young’s inequality for convolutions to show that $\|\omega^\ell\|_p \leq 2\|\omega\|_p$ and use the definition of $\mathcal{H}_p$ to show that

\[
\|h'(\omega)\|_{p/(p-1)} \leq 2^{(p-1)/p} \max \left\{ M_h, C \|\omega\|_{p-1} \right\}.
\]

From the above estimates conclude that

\[
\lim_{\ell \to 0} \|h(\bar{\omega}^\ell) - h(\omega)\|_1 = 0.
\]

**Remark:** For the proof that $\lim_{\ell \to 0} \|\bar{\omega}^\ell - \omega\|_p = 0$, see E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Chapter III, Section 2.2, Theorem 2.