(E) Does Any of this Apply to Navier-Stokes Turbulence?

In this section we consider — briefly — how any of these results might carry over to real fluid turbulence, governed by the Navier-Stokes equation.

The phenomenon of spontaneous stochasticity must be presumed to carry over to the Navier-Stokes solutions in the limit as $\nu \to 0$ or $Re \to \infty$. In fact, we have seen that this phenomenon is just a manifestation of Richardson diffusion and the fact that initial particle separation $\Delta_0$ is “forgotten” at long times. This latter effect is verified in some recent numerical studies, e.g.


who observed long-time independence of noise amplitude in dispersion of stochastic Lagrangian trajectories in a DNS of hydrodynamic turbulence at $Re_\lambda = 433$:

![Diagram of mean dispersion of particle pairs](image)

**FIG. 1.** (Color) Mean dispersion of particle pairs: (a) backward dispersion, (b) forward dispersion. The backward-in-time results are plotted against $t' = t_f - t$ and all quantities are nondimensionalized with viscous units (see text). The $Pr = 1$ results are plotted with solid lines, $Pr = 0.1$ results with dashed lines. We color code the lines with blue for raw data (including error bars), green for short-time molecular diffusion, and red for long-time Richardson diffusion.
A similar “forgetfulness” has been observed for dispersion of deterministic Lagrangian trajectories in the previously mentioned study of Bitane et al. (2013) using DNS up to $Re_\lambda = 730$:

These observations confirm that the basic mechanism of “spontaneous stochasticity” exists in hydrodynamic turbulence. One can anticipate that the phenomenon will eventually be observed also in laboratory experiments and that its detailed characteristics will be more fully elucidated.

However, spontaneous stochasticity raises a number of serious puzzles for the traditional Lagrangian picture of 3D energy cascade, based on G. I. Taylor’s idea of vortex line-stretching. Can there be material lies at all, if Lagrangian particle trajectories are stochastic? If the notion of a material loop still makes sense, then does the Kelvin Theorem hold in any approximate sense? These questions are still not definitively answered. However, let us offer some reasonable conjectures and partial results. For more details, see


In the first place we believe that there should also be a turbulent diffusion process of material loops. That is, there should be a nontrivial probability distribution $P_u(C,t|C_0,t_0)$ for a loop $C_0$ at time $t_0$ to flow into a loop $C$ at the later (or earlier) time $t$. The picture is:

(This clearly requires a careful limiting procedure, e.g. smoothing the velocity, $u \to \bar{u}_\ell$, and then taking the limit $\ell \to 0$ after evolving $t_0 \to t$. Otherwise, taking $\ell \to 0$ first before $t_0 \to t$, one may expect the loop to “explode” into a disconnect cloud of particles!) This is one possible answer to the problem of the existence of material loops: they exist but become stochastic. Even in the Kraichnan model, where spontaneous stochasticity of points is rigorously established, there are presently no results about higher-dimensional objects such as loops.

What about the Kelvin Theorem? Eyink (2006, 2007) offered the following conjecture, that circulations are conserved on average taking the expectation over the stochastic ensemble of material loops:

$$\Gamma(C,t) = \int P_u(dC_0,t_0|C,t) \Gamma(C_0,t_0), \ t > t_0$$

As explained more fully in those papers, this is the exact formal analog of equation ($**$) for the passive scalar. Furthermore, the subsequent work of Constantin-Iyer (2008) showed that exactly this result is true for smooth solutions of incompressible Navier-Stokes in their stochastic.
formulation. The remaining—and apparently quite hard—problem is to show that the property remains meaningful and valid in the limit $\nu \to 0$. Among many difficult issues, loops advected by turbulence are expected to become fractal and non-rectifiable (i.e. infinitely long!) See:


If this conjecture is correct, it still remains to connect this “generalized Kelvin Theorem” to energy cascade in a precise way. We believe however that this is a key problem in theoretical turbulence of the 21st century. It is interesting in this context to quote two famous 20th century scientists, who struggled hard with these issues. The first is G. I. Taylor himself who, in a 1975 interview, discussed some of his early work on turbulence:

“Though in 1937 I had realized the equivalence of the correlation description of turbulence and the spectrum description, my idea of the dynamics was directed to trying to connect the rate of increase of mean-square vorticity with dispersion, because if two neighboring points on a vortex line are separating the vorticity is increasing, and of course the rate of dissipation of energy is increasing.”

But Taylor concludes:

“However, I did not see how to turn this idea into a mathematical description which could form the basis of a theory and could predict things that could be verified or disproved experimentally.”

Nevertheless, Taylor’s ideas have proved extremely influential and are now generally believed to be true, despite the many difficulties surrounding them. Let us quote from the famous undergraduate textbook of R. P. Feynman, who, in section 41-5 “The limit of zero viscosity” discusses the infinite Reynolds-number limit of turbulent flow:

“You may be wondering, ‘What is the fine-grain turbulence and how does it maintain itself? How can the vorticity which is made somewhere at the edge of the cylinder generate so much noise in the background?’ The answer is again interesting. Vorticity has a tendency to amplify itself. If we forget for a moment about the diffusion of vorticity which causes a loss, the laws of flow say (as we have seen) that the vortex lies are carried along with the fluid, at the velocity $v$. We can imagine a certain number of lines of $\Omega$ which are being distorted and twisted by the complicated flow pattern of $v$. This pulls the lines closer together and mixes them all up. Lines that were simple before will get knotted and pulled close together. They will be longer and tighter together. The strength of the vorticity will increase and its irregularities - the pluses and minuses - will, in general, increase. So the magnitude of vorticity in three dimensions increases as we twist the fluid about.”


These views seem very plausible and probably contain a germ of truth. However, we have seen that, upon reflection, the matter is very subtle and difficult. It is not even clear that a simple, natural idea — that “the strength of the vorticity will increase ...[as vortex lines get knotted and pulled together]” — is true, at least in the most naive sense. However, experiments and simulations are getting better. Also, our mathematical tools have been considerably sharpened. Perhaps the 21st century shall be the one in which all those questions are finally, answered!