**Homework No.8, 553.793, Due April 29, 2022.**

1. This problem studies the decreasing function $h_p = d\zeta_p/dp$. As we shall see later in studying the multifractal formalism, $h_p$ has an interpretation as the Hölder singularity of the velocity field which makes the dominant contribution to the inertial-range scaling of the structure function of order $p$.

(a) Show that the concave function $\zeta_p$ satisfies the inequality

$$\frac{\zeta_{p+q} - \zeta_p}{q} \leq \frac{\zeta_p - \zeta_q}{p-q}.$$

for $0 \leq q \leq p$. *Hint:* $p$ lies between $q$ and $p+q$.

(b) Use part (a) to show that

$$2\sigma_{2p} - \sigma_p \leq h_p \leq \sigma_p$$

where $\sigma_p = \zeta_p/p$.

(c) Conclude that $\lim_{p\to\infty} h_p = h_{\min}$.

2. In this problem we shall consider several theoretical predictions for the scaling exponents $\zeta_p$ of the longitudinal velocity structure functions. These are the predictions by the lognormal model of A. N. Kolmogorov, J. Fluid Mech. **13** 82 (1962)

$$(K62) \qquad \zeta_p = \frac{p}{3} - \frac{\mu}{18}p(p-3)$$

with $\mu = 0.25$, the log-Poisson model of Z.-S. She and E. Lévêque, Phys. Rev. Lett. **72** 336 (1994)

$$(SL) \qquad \zeta_p = \frac{p}{9} + 2\left[1 - \left(\frac{2}{3}\right)^{p/3}\right]$$

and the "mean-field theory" of V. Yakhot, Phys. Rev. E **63** 026307 (2001)

$$(MF) \qquad \zeta_p = \frac{ap}{b+cp}$$

with $a = 0.185, b = 0.473$ and $c = 0.0275$.

(a) For each of these theoretical models for $\zeta_p$ calculate the prediction for $h_p$.

(b) Plot the three models for $\zeta_p$ over the range $0 \leq p \leq 10$ and then do likewise for the three predictions of $h_p$. Which of these better distinguishes the different models?

(c) Calculate the value $h_{\min}$ predicted by each of the three models.

3. This problem studies the scaling exponents of the velocity structure-functions empirically, using data from a $1024^3$ simulation in a periodic box of forced, steady-state turbulence stored in an online database at the Johns Hopkins University:

`http://turbulence.pha.jhu.edu/docs/README-isotropic.pdf`

The velocity fields in 192 vertical "cuts" through this database have been downloaded and made available at the course website in a zipped directory:

`http://www.ams.jhu.edu/~eyink/Turbulence/TURBDATA.zip`

When uncompressed, the directory "DATA" contains folders "RUN1", ...,"RUN192" storing velocity-field data harvested along the cuts. These can be easily read in Matlab, for example, with the command

```
vel=load(sprintf('DATA/RUN%d/velocity.dat',k));
```

for `k=1:192` with the resulting array `vel` containing the z-position in the first column, and the x-,y- and z-components of the velocity field in the second, third, and fourth columns, respectively. The array created by the Matlab command

```
dv=circshift(vel,1-ii)-vel;
```

for `ii=1:512` then creates the velocity-differences with increments `rr=ii*2*pi/1024` directed along the z-axis.

(a) Using this data, calculate numerically the absolute velocity-structure functions

$$S_p^X(r) = \langle |\delta v^X(\mathbf{r})|^p \rangle,$$

for $p = 1, 2, ..., 10$ and $X = L$ (longitudinal), $C = N$ (transverse) and plot them log-log. By a least-squares fit in log-log over the two-octave range $r = (0.0561)L$ to $r = (0.2045)L$, estimate the constants $C_p^X$ and scaling exponents $\zeta_p^X$ in the inertial-range scaling laws

$$S_p^X(r) \sim C_p^X u_{rms}^p (r/L)^{\zeta_p^X}.$$

Based upon your plots, what is the largest value of $p$ for which you believe that your results for exponents and amplitudes are reliable?

(b) Plot your results for $\zeta_p^L$ and $\zeta_p^N$ versus $p$, for $p = 0, 1, ..., 6$. Which velocity-differences are more intermittent, the longitudinal or the transverse?

(c) Repeat the plot of longitudinal exponents $\zeta_p^L$ versus $p$, for $p = 0, 1, ..., 6$ and plot also the predictions of K41 ($\zeta_p^L = p/3$) and of the three models in Problem #2 (log-normal, log-Poisson, mean-field). Are the predictions distinguishable by the data?

4. For one-dimensional decaying Burgers equation

$$\partial_t u + u\partial_x u = \nu\partial_x^2 u,$$

derive the following balance relation for the "point-split" energy density

$$\partial_t\left(\frac{1}{2}uu'\right) + \partial_x\left[\frac{1}{4}u'u^2 + \frac{1}{12}u'^3 - \nu\partial_x\left(\frac{1}{2}uu'\right)\right] = \partial_r\left(\frac{1}{12}\delta u^3\right) - \nu(\partial_x u)(\partial_x u'),$$

where $u = u(x,t), u' = u(x+r,t)$ and $\delta u = u' - u = u(x+r,t) - u(x,t)$.