Homework No.6, 550.694, Due April 30, 2008.

1. (a) Derive the RANS time-evolution equation for the Reynolds stress $\tau_{ij} = \overline{u_i' u_j'}$. By taking one-half of the trace, derive the evolution equation of the turbulent kinetic energy $k = (1/2)\overline{q^2}$.

(b) Specialize the evolution equation for k obtained in (a) to fully-developed turbulent flow in a channel between infinite, parallel planes. Justify carefully all of the simplications in the equations.

2. This problem concerns the wall-normal variation (y-variable dependence) of the mean pressure in turbulent channel flow.

(a) Show that the mean pressure can be written as

$$\overline{p}(x,y) = \overline{p}_0(x) - \overline{v'^2}(y),$$

where $\overline{p}_0(x) = (const.) - \gamma x$ is the mean pressure-gradient at the wall.

(b) Use the result

$$\frac{\partial \overline{p_*}}{\partial y} = -\overline{\Sigma}_{xz}$$

with $p_* = p + (1/2)|\mathbf{u}|^2$ to show that

$$-\frac{\partial}{\partial y}(\overline{p} + k) = \overline{u'\omega_z' - w'\omega_x'}.$$

3. (a) Use the results in Batchelor, *An Introduction to Fluid Dynamics*, (Cambridge, 1967), Appendix 2, in order to derive the Navier-Stokes equation in cylindrical coordinates, in the form that it appeared in the classnotes. Use this result to derive the RANS equations for the three components of mean velocity in pipe-flow, using cylindrical coordinates.

(b) Write the Helmholtz equation $\partial_t \boldsymbol{\omega} = \boldsymbol{\nabla}(\mathbf{u}\times\boldsymbol{\omega} - \nu\boldsymbol{\nabla}\times\boldsymbol{\omega})$ in cylindrical coordinates. Use this result to derive the RANS equations for the three components of mean vorticity in pipe-flow, using cylindrical coordinates.

4. (a) The *bulk velocity* $\overline{u}_m$ in channel-flow is defined by a cross-sectional average:

$$\overline{u}_m = \frac{1}{2h}\int_0^{2h} \overline{u}(y)\, dy.$$

Use the velocity-defect law to argue that

$$\frac{\overline{u}_m - \overline{u}_c}{u_*} = C_1$$

where $\overline{u}_c = \overline{u}(h)$ is the mean velocity at the centerplane and C_1 is a constant independent of the Reynolds-number.

(b) The *coefficient of friction* or *friction factor* is defined as the ratio

$$\gamma = \frac{\tau_0}{\frac{1}{2}\overline{u}_m^2} = 2\left(\frac{u_*}{\overline{u}_m}\right)^2$$

where τ_0 is the shear-stress at the wall. Use part (a) to show that Prandtl's logarithmic friction law can be rewritten as

$$\sqrt{\frac{2}{\gamma}} = \frac{1}{\kappa}\ln(Re\sqrt{\gamma}) + C_2$$

for the bulk Reynolds number $Re = \overline{u}_m h/\nu$ and an appropriate constant C_2.