Problem 1 (a) By definition

\[ \frac{\partial}{\partial x_i} f_\varepsilon(x) = \lim_{\varepsilon \to 0} \frac{f_\varepsilon(x + \varepsilon e_i) - f_\varepsilon(x)}{\varepsilon} \]

\[ = \lim_{\varepsilon \to 0} \int d^3r \, G_\varepsilon(r) \left[ \frac{f(x + \varepsilon e_i + r) - f(x + r)}{\varepsilon} \right] \]

using the definition of $f_\varepsilon$. Under reasonable assumptions (e.g., if $f$ is uniformly differentiable in a region of radius $\approx \varepsilon$ around $x$), one can take the limit inside the integral to obtain

\[ \frac{\partial}{\partial x_i} f_\varepsilon(x) = \int d^3r \, G_\varepsilon(r) \lim_{\varepsilon \to 0} \left[ \frac{f(x + r + \varepsilon e_i) - f(x + r)}{\varepsilon} \right] \]

\[ = \int d^3r \, G_\varepsilon(r)(\partial_i f)(x + r). \]

(b) Again using the definition of $f_\varepsilon$,

\[ \int d^d x \, f_\varepsilon(x) = \int d^d x \, \int d^3r \, G_\varepsilon(r) f(x + r) \]

\[ = \int d^d x \, \int d^d x' \, G_\varepsilon(x - x') f(x') \quad \forall x \equiv x' \]

Under modest assumptions one can commute the $x$ and $x'$-integrals. For example, this is possible if absolute integrability holds

\[ \int d^d x \, \int d^d x' \, |G_\varepsilon(x - x')| f(x') < \infty, \]

which will generally be true. In that case

\[ \int d^d x \, f_\varepsilon(x) = \int d^d x' \, f(x') \left[ \int d^d x \, G_\varepsilon(x - x') \right]. \]

However, $\int d^d x \, G_\varepsilon(x - x') = 1$ by the normalization condition on $G$. Thus,

\[ \int d^d x \, f_\varepsilon(x) = \int d^d x' \, f(x'). \]
Problem 2. We shall use

\[(f * g)(x) = \int \mathbb{R}^d f(r) g(x-r)\]

\[= \lim_{\Delta r \to 0} \sum_{i=1}^{n} (\Delta r)^d f(r_i) g(x-r_i)\]

where \(\sum_{i=1}^{n} (\Delta r)^d f(r_i) g(x-r_i)\) is a finite Riemann sum which approximates the integral. Now

\[\| \sum_{i=1}^{n} (\Delta r)^d f(r_i) g(\cdot - r_i) \|_p \leq \sum_{i=1}^{n} (\Delta r)^d |f(r_i)| \cdot \|g(\cdot - r_i)\|_p\]

by the triangle inequality. Since the \(L^p\)-norm itself is shift-invariant, \(\|g(\cdot - r_i)\|_p = \|g\|_p\) for all \(i\), and thus

\[\| \sum_{i=1}^{n} (\Delta r)^d f(r_i) g(\cdot - r_i) \|_p \leq \left[ \sum_{i=1}^{n} (\Delta r)^d |f(r_i)| \right] \cdot \|g\|_p\]

Taking the limit \(\Delta r \to 0\) and using (or assuming) that the Riemann sum converges to the integral in the \(L^p\)-norm sense,

\[\|f * g\|_p = \lim_{\Delta r \to 0} \| \sum_{i=1}^{n} f(r_i) g(\cdot - r_i) \|_p\]

\[\leq \lim_{\Delta r \to 0} \| \sum_{i=1}^{n} f(r_i) g(\cdot - r_i) \|_p\]

(cont'd)
\[ \lim_{\Delta r \to 0} \left( \sum_{i=1}^{n} (\Delta r)^d \cdot |f(x_i)| \right) \cdot \| g \|_p = \int \| f \|_1 \cdot \| g \|_p \]

QED

NOTE: This is not, of course, a rigorous argument according to the standards of mathematics. For example, we assumed without any proof that

\[ \lim \| R_n \| = \lim \| R_n \|_p \]

where \( R_n \) is the Riemann sum with \( N \) points. Also, we should use the more general notion of the Lebesgue integral, not the Riemann integral at all. One way to derive the inequality rigorously is to derive first the special cases

\[ \| f \cdot g \|_1 \leq \| f \|_1 \cdot \| g \|_1 \]

and

\[ \| f \cdot g \|_\infty \leq \| f \|_1 \cdot \| g \|_\infty \]

and then to use a general method called "Riesz-Thorin interpolation" to deduce the general case for \( 1 \leq p \leq \infty \). (In fact, the general "Young inequality for convolutions" can be obtained by a second application of Riesz-Thorin interpolation.) The above "proof" should give however an intuitive understanding of the result, as a continuous generalization of the triangle inequality.
Problem 3. (a) Note that

\[ E(X_1), E(X_1X_2), E(X_1X_2X_3) \]

are invariant under permutation of their arguments, by commutativity
of ordinary multiplication. Furthermore,

\[ C(X_1, X_2) = E(X_1X_2) - E(X_1) \cdot E(X_2) \]

is permutation invariant, since both \( E(X_1X_2) \) and \( E(X_1), E(X_2) \) are so.

The same argument then applies to

\[ C(X_1, X_2, X_3) = E(X_1X_2X_3) \]

\[- \left[ E(X_1) C(X_2, X_3) + E(X_2) C(X_1, X_3) + E(X_3) C(X_1, X_2) \right] \]

\[- E(X_1) \cdot E(X_2) \cdot E(X_3), \]

since each of the three terms on the right-hand side are now
seen to be permutation invariant.

**NOTE:** This argument can be generalized to an inductive proof
that all cumulants \( C(X_1, \ldots, X_n) \) are permutation invariant
for every \( n \). However, there are easier proofs!
\( C(x_1 + a_1) = E(x_1 + a_1) \)
\[ = \int (x_1 + a_1) dP = \int x_1 dP + a_1 \int dP = E(x_1) + a_1 \]

\[ C(x_1 + a_1, x_2 + a_2) = E((x_1 + a_1)(x_2 + a_2)) - E(x_1 + a_1)E(x_2 + a_2) \]
\[ = E(x_1 x_2 + a_1 x_2 + a_2 x_1 + a_1 a_2) \]
\[ - \left[E(x_1) + a_1 \right] \left[ E(x_2) + a_2 \right] \]
\[ = \left[E(x_1 x_2) + a_1 E(x_2) + a_2 E(x_1) + a_1 a_2 \right] \]
\[ - \left[E(x_1) E(x_2) + a_1 E(x_2) + a_2 E(x_1) + a_1 a_2 \right] \]
\[ = E(x_1 x_2) - E(x_1) E(x_2) = C(x_1, x_2) \]

For \( n = 3 \), we use
\[ C(x_1 + a_1, x_2 + a_2, x_3 + a_3) = E((x_1 + a_1)(x_2 + a_2)(x_3 + a_3)) \]
\[ - E(x_1 + a_1) C(x_2, x_3) - E(x_2 + a_2) C(x_1, x_3) - E(x_3 + a_3) C(x_1, x_2) \]
\[ - E(x_1 + a_1) E(x_2 + a_2) E(x_3 + a_3) \]
Then, we use

\[ E((X_1 + a_1)(X_2 + a_2)(X_3 + a_3)) \]

\[ = E(X_1X_2X_3) + a_1 E(X_2X_3) + a_2 E(X_1X_3) + a_3 E(X_1X_2) \]

\[ + a_1a_2 E(X_3) + a_1a_3 E(X_2) + a_2a_3 E(X_1) + a_1a_2a_3 \]

and

\[ E(X_1 + a_1) E(X_2 + a_2) E(X_3 + a_3) \]

\[ = E(X_1) E(X_2) E(X_3) + a_1 E(X_1) E(X_2) + a_2 E(X_1) E(X_3) + a_3 E(X_1) E(X_2) \]

\[ + a_1a_2 E(X_3) + a_1a_3 E(X_2) + a_2a_3 E(X_1) + a_1a_2a_3 \]

to set

\[ E((X_1 + a_1)(X_2 + a_2)(X_3 + a_3)) - E(X_1 + a_1) E(X_2 + a_2) E(X_3 + a_3) \]

\[ = E(X_1X_2X_3) - E(X_1) E(X_2) E(X_3) \]

\[ + a_1 \text{C}(X_1, X_2) + a_2 \text{C}(X_1, X_3) + a_3 \text{C}(X_1, X_2), \]

Finally, we obtain from (1) that

\[ \text{C}(X_1 + a_1, X_2 + a_2, X_3 + a_3) = E(X_1X_2X_3) - E(X_1) \text{C}(X_2, X_3) \]

\[ - E(X_2) \text{C}(X_1, X_3) - E(X_3) \text{C}(X_1, X_2) \]

\[ - E(X_1) E(X_2) E(X_3) = \text{C}(X_1, X_2, X_3), \]

\( \Box \)
Problem 4, (a)

\[ \hat{\nabla} \bar{f}(x) = \int d^d r \hat{G}(r) \bar{f}(x + r) \]

\[ = \int d^d r \hat{G}(r) \left[ \int d^d r' \bar{G}(r') \bar{f}(x + r + r') \right] \]

Now, set

\[ r'' = r + r' \]

and write

\[ \hat{\nabla} \bar{f}(x) = \int d^d r \hat{G}(r) \int d^d r'' \bar{G}(r'' - r) \bar{f}(x + r'') \]

\[ = \int d^d r'' \left[ \int d^d r \hat{G}(r) \bar{G}(r'' - r) \right] \bar{f}(x + r'') \]

\[ = \left( \hat{\nabla} \ast \bar{G} \right)(r'') \]

by interchanging integrals and using the definition of convolution.

Thus,

\[ \hat{\nabla} \bar{G}(r'') = \left( \hat{\nabla} \ast \bar{G} \right)(r'') \]

QED
\[ (b) \quad \bar{\sigma}(f,g) = \bar{\tau}_{fg} - \bar{\tau}_{f \bar{g}} \]
\[ = \left[ \frac{\bar{\tau}_{fg}}{fg} - \frac{\bar{\tau}_{f \bar{g}}}{f \bar{g}} \right] + \left[ \frac{\bar{\tau}_{f \bar{g}}}{f \bar{g}} - \frac{\bar{\tau}_{fg}}{fg} \right] \]
\[ = \left[ \frac{\bar{\tau}_{fg}}{fg} - \frac{\bar{\tau}_{f \bar{g}}}{f \bar{g}} \right] + \left[ \frac{\bar{\tau}_{fg}}{fg} - \frac{\bar{\tau}_{f \bar{g}}}{f \bar{g}} \right] \]
\[ = \bar{\tau}(f,g) + \bar{\tau}(\bar{f}, \bar{g}) \quad \text{QED} \]

**Problem 5.** By setting \( i = j \) and dividing by 2 in the evolution equation for stress, one obtains, with \( k_i = \frac{1}{2} T_{ii} \) (no summation on \( i \))

\[ \partial_t k_i + \partial_k \left( \frac{1}{2} J_{ii} u \right) = - \bar{u}_i, \bar{u}_i T_{ii} + \tau(p, S_{ii}) \cdot n \left( u_{i+1}, u_{i-1}, \bar{u}_i, \bar{u}_i \right) + \tau(u_i, f_i) \]

The pressure-strain term

\[ \tau(p, S_{ii}) = \tau(p, \frac{\partial u_i}{\partial x_i}) \] (no summation on \( i \))

cancels when summed over \( i \)

\[ \sum_i \tau(p, S_{ii}) = 0. \]
Thus, there is no net gain or loss of small-scale kinetic energy \( \sum_i \Delta E_i \) by the pressure-strain term, which, instead, simply transfers energy between different components of the velocity. In an anisotropic flow such as a shear flow, this term is usually responsible for the transfer of energy from the "minor component" — which absorbs energy directly from the mean shear — and into the "major components." See Tanneau & Lundley, p. 74.

**Problem 6.** We shall prove by induction that, for \( r < 1 \),

\[
\frac{\partial^n}{\partial r_1 \ldots \partial r_n} G(r) = \exp \left( -\frac{1}{1-r^2} \right) \frac{P(r)}{(1-r^2)^n}
\]

where \( P(r) \) is a polynomial in the coordinates \( r_1, \ldots, r_n \).

First, for \( n = 1 \)

\[
\frac{\partial}{\partial r_i} G(r) = \exp \left( -\frac{1}{1-r^2} \right) \frac{-2r_i}{(1-r^2)^2}
\]

by the chain rule. This gives the result with \( P(r) = -2r_i \).
Now assume the result for \( n \) and consider

\[
\frac{d^{n+1}}{dr_{n+1} \ldots dr_1} G(r) = \frac{2}{\partial r_{n+1}} \left[ \exp \left( -\frac{1}{1-r^2} \right) \frac{P(r)}{(1-r^2)^{2n}} \right]
\]

\[= \exp \left( \frac{-1}{1-r^2} \right) \left[ \frac{-2r_{n+1} P(r)}{(1-r^2)^{2(n+1)}} + \frac{2}{\partial r_{n+1}} \frac{P(r)}{(1-r^2)^{2n}} \right]
\]

\[+ P(r) \frac{4nr_{n+1}}{(1-r^2)^{2n+1}} \]

\[= \exp \left( \frac{-1}{1-r^2} \right) \frac{P(r)}{(1-r^2)^{2(n+1)}} \]

with

\[
\bar{P}(r) = -2r_{n+1} P(r) + (1-r^2) \frac{2}{\partial r_{n+1}} P(r) + 4nr_{n+1} (1-r^2) P(r)
\]

a polynomial in the variables. This gives the stated result, by induction.

We now consider the limit from below

\[
\lim_{r \to 1-} \frac{d^{n}}{dr_{n} \ldots dr_1} G(r) = \lim_{r \to 1-} \exp \left( -\frac{1}{1-r^2} \right) \frac{P(r)}{(1-r^2)^{2n}}
\]
One can see that the denominators go to zero

\[(1-r^2)^{2n} \to 0\]

but the numerators go to zero even faster

\[\exp\left(\frac{-1}{1-r^2}\right) \to 0\] faster than any polynomial

This may be shown by l'Hôpital's rule (since the derivatives of \((1-r^2)^{2n}\) becomes constant, but the derivatives of the numerator becomes \(\exp\left(\frac{-1}{1-r^2}\right)\) times a polynomial.)

Another approach is to take logarithms and to note that

\[
\ln\left[ \exp\left(\frac{-1}{1-r^2}\right) \frac{P(r)}{(1-r^2)^{2n}} \right]
\]

\[= \frac{-1}{1-r^2} + \ln P(r) - 2n \ln (1-r^2)\]

\[\to -\infty \text{ as } r \to 1^-\]

Since the term \(\frac{-1}{1-r^2}\) is the largest in magnitude, from all of these arguments, we see that
\[
\lim_{r \to 1^- \partial r_1 \ldots \partial r_n} G(r) = 0
\]

for all integers \( n \). Thus, the definition

\[
G(r) = \begin{cases} 
\exp \left( \frac{-1}{1-r^2} \right) & r < 1 \\
0 & r \geq 1 
\end{cases}
\]

has all \( n \)th derivatives continuous at \( r = 1 \). Clearly, \( G(r) \) is \( C^\infty \) for \( r < 1 \) and \( r \geq 1 \). Thus, we have proved that \( G(r) \) is \( C^\infty \) for all values of \( r \).

\[QED\]
Problem 7. (a) Observing as in the classnotes that

\[ 2 \nu (\nabla \cdot \mathbf{v}) = \nu \nabla_j (\partial_i u_j + \partial_j u_i) \]
\[ = \nu \left( 0 + \partial_j^2 u_i \right) = \nu (\Delta \mathbf{v}) \cdot \mathbf{i} \]

we see that the two forms of the momentum equation

\[ \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla \mathbf{p} + \nu \Delta \mathbf{v} \]

and

\[ \partial_t \mathbf{v} + \nabla \cdot (\nu \mathbf{v} + p \mathbf{I} - 2\nu \mathbf{S}) = 0 \]

are equivalent for smooth solutions with the condition \( \nabla \cdot \mathbf{v} = 0 \). These two equations thus also imply the kinetic energy balance equation as discussed in the classnotes

\[ \partial_t \left( \frac{1}{2} |\mathbf{v}|^2 \right) + \nabla \cdot \left( \left( \frac{1}{2} |\mathbf{v}|^2 + p \right) \mathbf{v} - 2\nu \mathbf{S} \cdot \mathbf{v} \right) = -2\nu |\mathbf{S}|^2 \]
\[ = -\varepsilon. \]

Writing the Fourier temperature equation in the form

\[ \partial_t (c_p T) + \nabla \cdot (c_p T \mathbf{v} - (k/k_p) \nabla T) = \varepsilon, \]

we see that the sum of the last two equations gives

\[ \partial_t \left( \frac{1}{2} |\mathbf{v}|^2 + c_p T \right) + \nabla \cdot \left[ \left( \frac{1}{2} |\mathbf{v}|^2 + c_p T + p \right) \mathbf{v} \right. \]
\[ \left. - 2\nu \mathbf{S} \cdot \mathbf{v} - (k/k_p) \nabla T \right] = 0 \]

in conservation form.
On the other hand, the conservation equation (3) for total energy per mass \( \frac{1}{2}|\mathbf{v}|^2 + c_p T \) minus the equation (1) for kinetic energy per mass gives the equation (2) which is equivalent to the temperature equation

\[
\eta T + (\mathbf{v} \cdot \nabla) T = \lambda + \Delta T + \frac{e}{c_p}.
\]

Thus, the two systems of equations are entirely equivalent.

(b) We can evaluate the term involving \( K \) as

\[
\nabla \cdot \left[ \frac{(K/p \nabla T)}{\epsilon} \right] = -\frac{1}{\epsilon p} \int d\mathbf{r} \left( \nabla G \right)_\epsilon(\mathbf{r}) K(\mathbf{x} + \mathbf{r}) \nabla T(\mathbf{x} + \mathbf{r})
\]

\[
= -\frac{1}{\epsilon p} \int d\mathbf{r} \sum_{\text{supp} G_\epsilon(\mathbf{r})} \sqrt{K(\mathbf{x} + \mathbf{r})}

\left( \nabla G \right)_\epsilon(\mathbf{r}) \times \sqrt{K(\mathbf{x} + \mathbf{r})} \nabla T(\mathbf{x} + \mathbf{r})
\]

so that applying Cauchy-Schwarz inequality gives

\[
\left| \nabla \cdot \left[ \frac{(K/p \nabla T)}{\epsilon} \right] \right| \leq \frac{1}{\epsilon p} \int d\mathbf{r} \sum_{\text{supp} G_\epsilon(\mathbf{r})} \sqrt{K(\mathbf{x} + \mathbf{r})}

\times \sqrt{\int d^3 r \left( \nabla G \right)_\epsilon(\mathbf{r})^2 K(\mathbf{x} + \mathbf{r}) |\nabla T(\mathbf{x} + \mathbf{r})|^2}
\]
If we assume that, for fixed \( l \),

\[
\int_{\text{supp} \mathcal{G}_l} d^d r \ k(x+r) \to 0
\]

with

\[
\int d^3 x \ \varphi(x) \ k(x) \ |\nabla T(x)|^2
\]

remaining finite for compactly-supported, \( C^\infty \) functions \( \varphi \),
then we see that, for fixed \( l \),

\[
|\nabla \cdot \left[ (k/p) \nabla T \right]_{\mathcal{G}_l}(x) | \to 0.
\]

In a similar fashion, the term

\[
\nabla \cdot \left[ 2(\mathbf{v} \cdot \mathbf{S}) \right]_l = -\frac{1}{l} \int d^d r \ (\nabla G)_l(r) \cdot \mathbf{v}(x+r)
\]

\[
\cdot \mathbf{S}(x+r) \mathbf{v}(x+r)
\]

\[
= -\frac{1}{l} \int d^d r \ \frac{1}{\text{supp} \mathcal{G}_l(r)} \sqrt{\mathbf{v}(x+r)^T \mathbf{v}(x+r)}
\]

\[
\times \sqrt{\mathbf{v}(x+r)^T \mathbf{S} : \mathbf{v}(x+r)} (d \cdot \mathcal{G})_l(r).
\]

Applying Cauchy–Schwartz again with the above factorization of the integrand yields
\[
\left| \nabla \cdot \left[ \frac{2(\mathbf{v} \cdot \mathbf{v})}{2} \mathbf{v} \right] (x) \right| \leq \frac{1}{\ell} \sqrt{\int_{\text{supp} G} \int d^d r \ |\mathbf{v}(x+r)|^2 \left( \mathbf{v}(x+r) \cdot \mathbf{v}(x+r) \right)}
\times \sqrt{\int d^d r \ |(\nabla G)(\mathbf{r})|^2 \left( \mathbf{v}(x+r) \cdot \mathbf{S}(x+r) \right)}^2.
\]

Thus, assuming that
\[
(\star) \quad \int d^d r \ \mathbf{v}(x+r) \cdot \mathbf{v}(x+r) \to 0 \quad \text{supp} G
\]
and that \[\int d^d x \ \mathbb{G}(x) \mathbf{v}(x) \cdot \mathbf{S}(x) \to \text{finite} \] remains finite for smooth, compactly supported test functions, we see that
\[
\left| \nabla \cdot \left[ \frac{2(\mathbf{v} \cdot \mathbf{v})}{2} \mathbf{v} \right] (x) \right| \to 0.
\]

Note that \((\star)\) follows because
\[
\int d^d r \ \mathbf{v}(x+r) \cdot \mathbf{v}(x+r) \to \left[ \sup_{\mathbf{r} \in \text{supp} G} \mathbf{v}(x+r) \right] \int d^d r \ |\mathbf{v}(x+r)|^2
\]
with total kinetic energy \[\int d^d r \ |\mathbf{v}(x+r)|^2 \to \text{finite} \quad \text{and supp} \mathbf{v}(x+r) \to 0, \quad \text{re} \text{ suppg} \]