Homework No.3, 550.694, Due March 7, 2008.

1. (a) Show that the gradient of the Lagrangian map $\mathbf{X}(\boldsymbol{\alpha}, t)$ satisfies

$$\frac{\partial}{\partial t}\boldsymbol{\nabla}_\alpha \mathbf{X} = \boldsymbol{\nabla}_\alpha \mathbf{X} \cdot \boldsymbol{\nabla}_x \mathbf{u}.$$

Use this result to show that Cauchy's vorticity formula $\boldsymbol{\omega}(\mathbf{X}(\boldsymbol{\alpha}, t), t) = \boldsymbol{\Omega}(\boldsymbol{\alpha}) \boldsymbol{\cdot} \boldsymbol{\nabla}_\alpha \mathbf{X}(\boldsymbol{\alpha}, t)$ provides an explicit integration of the 3D Euler equation $D_t \boldsymbol{\omega} = \boldsymbol{\omega} \boldsymbol{\cdot} \boldsymbol{\nabla}_x \mathbf{u}$.

(b) The *Weber velocity variable*

$$\mathbf{w}(\boldsymbol{\alpha}, t) \equiv \boldsymbol{\nabla}_\alpha \mathbf{X}(\boldsymbol{\alpha}, t) \boldsymbol{\cdot} \mathbf{v}(\boldsymbol{\alpha}, t)$$

is closely related to the Cauchy invariant. Establish the so-called Weber formulation of the 3D Euler equation:

$$\frac{\partial}{\partial t}\mathbf{w} = \boldsymbol{\nabla}_\alpha \left[\frac{1}{2}|\mathbf{v}|^2 - p_L\right],$$

where note that $p_L(\boldsymbol{\alpha}, t) = p(\mathbf{X}(\boldsymbol{\alpha}, t), t)$ is the Lagrangian pressure.

(c) If C is any fixed loop in the label space, show that

$$\oint_C d\boldsymbol{\alpha} \boldsymbol{\cdot} \mathbf{w}(\boldsymbol{\alpha}, t) = \oint_{C(t)} d\mathbf{x} \boldsymbol{\cdot} \mathbf{u}(\mathbf{x}, t)$$

where $C(t)$ is the image of C under the Lagrangian flow $\mathbf{X}(\boldsymbol{\alpha}, t)$. Then use the result in part (b) to give another proof of the Kelvin circulation theorem.

(d) Show that Cauchy's vorticity invariant is the curl of Weber's velocity variable:

$$\boldsymbol{\Omega}(\boldsymbol{\alpha}) = \boldsymbol{\nabla}_\alpha \times \mathbf{w}(\boldsymbol{\alpha}, t).$$

Hint: Define $\boldsymbol{\Omega}^*(\boldsymbol{\alpha}) \equiv \boldsymbol{\nabla}_\alpha \times \mathbf{w}(\boldsymbol{\alpha}, t)$ and then calculate $\boldsymbol{\Omega}^*(\boldsymbol{\alpha}) \boldsymbol{\cdot} \boldsymbol{\nabla}_\alpha \mathbf{X}(\boldsymbol{\alpha}, t)$. You will find useful the result

$$\epsilon_{ijk} \frac{\partial X_l}{\partial \alpha_i} \frac{\partial X_m}{\partial \alpha_j} \frac{\partial X_n}{\partial \alpha_k} = \epsilon_{lmn},$$

which you should show follows from the Jacobian, $\partial(X_1, X_2, X_3)/\partial(\alpha_1, \alpha_2, \alpha_3) = 1$.

2. (a) Show that

$$\int_0^t ds \int_0^t ds' \, F\left(s'-s, \frac{s'+s}{2}\right) = \left[\int_0^t d\tau \int_{\tau/2}^{t-\tau/2} dT + \int_{-t}^0 d\tau \int_{-\tau/2}^{t+\tau/2} dT\right] f(\tau, T)$$

with $\tau = s' - s$, $T = (s'+s)/2$.

(b) Use part (a) to show that, if the statistics of the Lagrangian velocity $\mathbf{v}(\boldsymbol{\alpha}, t)$ are stationary in time, then

$$\frac{\langle |\delta \mathbf{X}(t)|^2 \rangle}{2t} = \int_0^t d\tau \left(1 - \frac{\tau}{t}\right) \langle \mathbf{v}(\tau) \boldsymbol{\cdot} \mathbf{v}(0) \rangle.$$

(c) Finally, show that

$$\lim_{t\to\infty} \langle |\delta \mathbf{X}(t)|^2 \rangle / 2t = D$$

with $D = \int_0^\infty d\tau \langle \mathbf{v}(\tau) \boldsymbol{\cdot} \mathbf{v}(0) \rangle$, if the velocity auto-correlation function $\langle \mathbf{v}(\tau) \boldsymbol{\cdot} \mathbf{v}(0) \rangle$ is (absolutely) integrable in time.

3. (a) Suppose that $\mathbf{u}(\mathbf{x}, t)$ for all times t is a smooth velocity field in a flow domain Ω that is incompressible and has no flow across the boundary $\partial\Omega$ of the domain: $\boldsymbol{\nabla} \boldsymbol{\cdot} \mathbf{u} = 0$ and $\hat{\mathbf{n}} \boldsymbol{\cdot} \mathbf{u}|_{\partial\Omega} = 0$. Explain why, for any function $F(\mathbf{x}, t)$,

$$\int_\Omega d^d x \, F(\mathbf{x}, t) = \int_\Omega d^d \alpha \, F(\mathbf{X}(\boldsymbol{\alpha}, t), t),$$

where $\mathbf{X}(\boldsymbol{\alpha}, t)$ is the Lagrangian flow map generated by $\mathbf{u}$.

(b) Now suppose that $\mathbf{u}(\mathbf{x}, t)$ is an incompressible velocity field on all of Euclidean space $\mathbb{R}^d$ which, for simplicity, is absolutely bounded: $|\mathbf{u}(\mathbf{x}, t)| \leq u_{\max} < \infty$. If Ω_n is a sequence of domains $\Omega_n \uparrow \mathbb{R}^d$, then explain why, for any fixed time t,

$$\lim_{n\to\infty} \frac{1}{|\Omega_n|} \int_{\Omega_n} d^d x \, F(\mathbf{x}, t) = \lim_{n\to\infty} \frac{1}{|\Omega_n|} \int_{\Omega_n} d^d \alpha \, F(\mathbf{X}(\boldsymbol{\alpha}, t), t),$$

where $|\Omega_n|$ is the d-dimensional volume of Ω_n.

(c) If statistics are defined by space-averaging, then show that the single-time, 1-point PDF's of Eulerian velocity $P(\mathbf{u}, t)$ and of Lagrangian velocity $P(\mathbf{v}, t)$ are identical under either of the assumptions in (a) and (b). It may be helpful to consider the Fourier transform, or so-called characteristic function,

$$\langle \exp(i\mathbf{k} \boldsymbol{\cdot} \mathbf{u}(\cdot, t)) \rangle = \int d^d u \, e^{i\mathbf{k}\boldsymbol{\cdot}\mathbf{u}} P(\mathbf{u}, t)$$

with average on the left over $\mathbf{x}$, and similarly for $P(\mathbf{v}, t)$.

4. In this problem we consider the solution of Richardson's equation for 2-particle relative diffusion in 3D spherical coordinates,

$$\partial_t P(\rho,t) = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2 K(\rho)\frac{\partial}{\partial\rho}P(\rho,t)\right),$$

with eddy-diffusivity $K(\rho) = k_0\langle\varepsilon\rangle^{1/3}\rho^{4/3}$.

(a) Define a new time-like variable $\tau = k_0\langle\varepsilon\rangle^{1/3}t$ and a time-dependent length-scale $L(\tau) = \tau^{3/2}$. Looking for solutions of the self-similar form

$$P(\rho,\tau) = \frac{1}{L^3(\tau)}Q\left(\frac{\rho}{L(\tau)}\right),$$

show that $Q(x)$ with $x = \rho/L(\tau)$ must satisfy the equation

$$\frac{9}{2}Q(x) + \frac{3}{2}xQ(x) + \frac{10}{3}x^{1/3}Q'(x) + x^{4/3}Q''(x) = 0.$$

(b) Making the substitution $y = \frac{3}{2}x^{2/3}$, show that the resulting equation has an exact solution $Q(y) = \exp(-(3/2)y)$. From this infer that

$$P(\rho,t) = \frac{A}{(k_0\langle\varepsilon\rangle^{1/3}t)^{9/2}}\exp\left[-\frac{9}{4}\frac{\rho^{2/3}}{k_0\langle\varepsilon\rangle^{1/3}t}\right]$$

is an exact solution of Richardson's equation for any constant A.

(c) Show that the value of the constant to satisfy the normalization condition

$$4\pi\int_0^\infty \rho^2 d\rho\, P(\rho,t) = 1$$

is given by $1/A = 4\pi(2/3)^8\Gamma(9/2)$ with $\Gamma(z) = \int_0^\infty dt\, t^{z-1}e^{-t}$. Explain why it is true, with this value of A, that $\lim_{t\to 0+} P(\rho,t) = \delta^3(\boldsymbol{\rho})$.

(d) Show for the solution in (b),(c) that

$$\langle\rho^2(t)\rangle \equiv 4\pi\int_0^\infty \rho^4 d\rho\, P(\rho,t) = g_0\langle\varepsilon\rangle t^3,$$

with the Richardson constant $g_0 = (1144/81)k_0^3$. *Hint:* Recall that $\Gamma(z+1) = z\Gamma(z)$.