Homework No.1, 550.694, Due February 22, 2008.

1. We consider some kinematic (instantaneous) properties of vortex tubes.

(a) A simple analytical model is provided by a rectilinear vortex tube with circular cross-section of radius R. This model has a 2D velocity in the plane perpendicular to the vortex axis given by a stream function (in cylindrical coordinates r, θ, z):

$$\psi = \begin{cases} \frac{1}{4}\omega_0(R^2 - r^2) & r < R \\ \frac{1}{2}\omega_0 R^2 \ln(R/r) & r > R \end{cases}$$

Calculate the velocity $\mathbf{u} = -\hat{\mathbf{z}}\times\boldsymbol{\nabla}\psi$ and vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\mathbf{u}$. In particular, verify there is constant vorticity $\omega_0\hat{\mathbf{z}}$ in the interior of the tube and zero vorticity outside.

(b) Show that the velocity field in (a) defines an exact stationary solution of the 3D incompressible Euler equations, with pressure p given by

$$p = C(t) - \begin{cases} \omega_0\psi + \frac{1}{2}|\mathbf{u}|^2 & r < R \\ \frac{1}{2}|\mathbf{u}|^2 & r > R \end{cases}$$

for any (time-dependent) constant $C(t)$. *Hint*: Use $(\mathbf{u}\boldsymbol{\cdot}\boldsymbol{\nabla})\mathbf{u} = \boldsymbol{\nabla}(\frac{1}{2}|\mathbf{u}|^2) - \mathbf{u}\times\boldsymbol{\omega}$.

(c) The subscale stress $\boldsymbol{\tau}_\ell = \tau_\ell(\mathbf{u},\mathbf{u})$ calculated for the velocity $\mathbf{u}$ of the vortex tube in (a) has the form

$$\boldsymbol{\tau}_\ell = \alpha_\ell(r)\hat{\mathbf{r}}\hat{\mathbf{r}} + \beta_\ell(r)\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}},$$

for some positive functions $\alpha_\ell(r)$, $\beta_\ell(r)$, when the filter kernel $G_\ell(r)$ is isotropic. Here $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ are unit vectors in the coordinate r, θ directions. Show that the subscale force $\mathbf{f}_\ell^S = -\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\tau}_\ell$ is given by

$$\mathbf{f}_\ell^S = \frac{\beta_\ell(r) - (r\alpha_\ell)'(r)}{r}\hat{\mathbf{r}}.$$

The functions $\alpha_\ell(r)$, $\beta_\ell(r)$ depend upon the precise filter kernel employed, but it will usually be the case that $\beta_\ell(r) \geq (r\alpha_\ell)'(r)$ [note that the latter becomes negative at large enough r.] Conclude that $\mathbf{f}_\ell^S$ is directed radially outward from the vortex axis.

(d) (**BONUS**) For a more general velocity field of the form $\mathbf{u}(r,\theta) = \gamma(r)\hat{\boldsymbol{\theta}}$, show that

$$\alpha_\ell(r) = \int_0^\infty r'dr' \int_0^{2\pi} d\theta'\, G_\ell(\rho')\gamma^2(r')\sin^2\theta'.$$

$$\beta_\ell(r) = \int_0^\infty r'dr' \int_0^{2\pi} d\theta'\, G_\ell(\rho')\gamma^2(r')\cos^2\theta' - \left[\int_0^\infty r'dr' \int_0^{2\pi} d\theta'\, G_\ell(\rho')\gamma(r')\cos\theta'\right]^2,$$

for $\rho' = \sqrt{r^2 + r'^2 - 2rr'\cos(\theta')}$. *Hint:* Consider $\boldsymbol{\tau}_\ell(r,\theta)$ at $\theta = 0$.

2. This problem concerns the 1956 *JFM* paper of R. Betchov.

(a) Suppose that $\mathbf{S}$ is the symmetric, traceless 3×3 strain matrix with eigenvalues $\alpha > \beta > \gamma$. Show that the strain-skewness

$$\mathrm{Tr}\,(\mathbf{S}^3) = 3\alpha\beta\gamma.$$

Hint: Use $\mathrm{Tr}\,(\mathbf{S}^3) = \alpha^3 + \beta^3 + \gamma^3$ and $\alpha + \beta + \gamma = 0$.

(b) Use part (a) to show that $\mathrm{Tr}\,(\mathbf{S}^3) < 0$ if and only if $\beta > 0$. Then use Betchov's identity, $-\langle \mathrm{Tr}\,(\mathbf{S}^3)\rangle = (1/4)\langle \boldsymbol{\omega}^\top \mathbf{S}\boldsymbol{\omega}\rangle$, to argue that mean vortex-stretching requires the strain matrix to have "typically" two extensive directions and one contractive.

(c) Prove the following generalization of Betchov's identity: if $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are solenoidal vector fields and $\langle\cdot\rangle$ is any homogeneous average, then

$$\langle a_{i,j} b_{j,k} c_{k,i}\rangle + \langle c_{i,j} b_{j,k} a_{k,i}\rangle = 0,$$

where $a_{i,j} = \partial a_i/\partial x_j$, etc.

3. This problem uses the "nonlinear model" $\boldsymbol{\tau}_\ell^{NL} = \frac{1}{3}C_2\ell^2\overline{\mathbf{D}}_\ell\overline{\mathbf{D}}_\ell^\top$, with $D_{i,j} = \partial u_i/\partial x_j$, to approximate *3D helicity flux*

$$\Lambda_\ell = -2\boldsymbol{\nabla}\overline{\boldsymbol{\omega}}_\ell{:}\boldsymbol{\tau}_\ell.$$

(a) Introducing symmetric part $\mathbf{R}$ and anti-symmetric part $\boldsymbol{\Xi}$ of the vorticity-gradient $(\boldsymbol{\nabla}\boldsymbol{\omega})_{ij} = \partial_i\omega_j$, with $\boldsymbol{\nabla}\boldsymbol{\omega} = \mathbf{R} - \boldsymbol{\Xi}$, show that $\Lambda_\ell = -2\overline{\mathbf{R}}_\ell{:}\boldsymbol{\tau}_\ell$. Then use the "nonlinear model" $\boldsymbol{\tau}_\ell^{NL}$ and the similar decomposition $\mathbf{D} = \mathbf{S} + \boldsymbol{\Omega}$ into symmetric and antisymmetric parts to derive

$$\Lambda_\ell^{NL} = \frac{2}{3}C_2\ell^2\left[-\overline{\mathbf{R}}_\ell{:}\overline{\mathbf{S}}_\ell^2 + 2\overline{\mathbf{R}}_\ell{:}(\overline{\mathbf{S}}_\ell\overline{\boldsymbol{\Omega}}_\ell) + \frac{1}{4}\overline{\boldsymbol{\omega}}_\ell\overline{\mathbf{R}}_\ell\overline{\boldsymbol{\omega}}_\ell\right].$$

Here $\mathbf{A}{:}\mathbf{B} = \mathrm{Tr}\,(\mathbf{A}^\top\mathbf{B}) = A_{ij}B_{ij}$ and $\Omega_{ij} = -(1/2)\epsilon_{ijk}\omega_k$.

(b) Show that

$$\overline{\boldsymbol{\omega}}_\ell^\top\overline{\mathbf{R}}_\ell\overline{\boldsymbol{\omega}}_\ell = \boldsymbol{\nabla}\boldsymbol{\cdot}\left[\frac{1}{2}|\overline{\boldsymbol{\omega}}_\ell|^2\overline{\boldsymbol{\omega}}_\ell\right].$$

Conclude that $\langle\overline{\boldsymbol{\omega}}_\ell^\top\overline{\mathbf{R}}_\ell\overline{\boldsymbol{\omega}}_\ell\rangle = 0$ for any homogeneous average.

(c) Use the generalized Betchov identity in Problem 2(c) with $\mathbf{b} = \overline{\boldsymbol{\omega}}_\ell$ and $\mathbf{a} = \mathbf{c} = \overline{\mathbf{u}}_\ell$ to show that

$$-\langle\overline{\mathbf{R}}_\ell{:}\overline{\mathbf{S}}_\ell^2\rangle = -2\langle\overline{\boldsymbol{\Xi}}_\ell{:}(\overline{\mathbf{S}}_\ell\overline{\boldsymbol{\Omega}}_\ell)\rangle + \frac{1}{4}\langle\overline{\boldsymbol{\omega}}_\ell\overline{\mathbf{R}}_\ell\overline{\boldsymbol{\omega}}_\ell\rangle$$

for any homogeneous average. Together with part (b), this gives furthermore that $-\langle\overline{\mathbf{R}}_\ell{:}\overline{\mathbf{S}}_\ell^2\rangle = -2\langle\overline{\boldsymbol{\Xi}}_\ell{:}(\overline{\mathbf{S}}_\ell\overline{\boldsymbol{\Omega}}_\ell)\rangle$.

(d) Combine all the results of parts (a),(b),(c) to conclude that

$$\langle \Lambda_\ell^{NL} \rangle = \frac{4}{3} C_2 \ell^2 \langle \boldsymbol{\nabla} \overline{\boldsymbol{\omega}}_\ell \boldsymbol{:} (\overline{\mathbf{S}}_\ell \overline{\boldsymbol{\Omega}}_\ell) \rangle.$$

4. This problem will consider the balance equation of the subscale enstrophy

$$\zeta_\ell = \frac{1}{2} \tau_\ell(\omega_i, \omega_i).$$

We use the Einstein convention of summation over repeated indices. Hereafter we shall also, for convenience, drop explicit reference to the subscript ℓ.

(a) Show that

$$\partial_j \tau(\omega_j, u_i) = \tau(\omega_j, u_{i,j}) = \tau(\omega_j, S_{ij}).$$

(b) Derive the balance equation

$$\begin{aligned} \partial_t \zeta + \partial_k & \left[\zeta \overline{u}_k + \frac{1}{2} \tau(\omega_i, \omega_i, u_k) - \nu \partial_k \zeta \right] \\ &= \tau(S_{ij}, \omega_i, \omega_j) + \tau(\omega_i, \omega_j) \overline{S}_{ij} + \tau(\omega_i, S_{ij}) \overline{\omega}_j \\ &\quad - \overline{\omega}_{i,j} \tau(\omega_i, u_j) - \nu \tau(\omega_{i,j}, \omega_{i,j}) \end{aligned}$$

This is analogous to equation (3.3.38) of Tennekes & Lumley (1972), derived with Reynolds averaging.

(c) Show that the equation in (b) is equivalent to

$$\begin{aligned} \partial_t \zeta + \partial_k & \left[\zeta \overline{u}_k + \frac{1}{2} \tau(\omega_i, \omega_i, u_k) + \tau(\omega_i, u_k) \overline{\omega}_i + \Sigma^*_{ki} \overline{\omega}_i - \nu \partial_k \zeta \right] \\ &= \left[\tau(S_{ij}, \omega_i, \omega_j) + \tau(\omega_i, \omega_j) \overline{S}_{ij} + 2\tau(\omega_i, S_{ij}) \overline{\omega}_j \right] \\ &\quad - \overline{\omega}_{i,j} \Sigma^*_{ji} - \nu \tau(\omega_{i,j}, \omega_{i,j}) \end{aligned}$$

where

$$\Sigma^*_{ij} = \tau(u_i, \omega_j) - \tau(\omega_j, u_i)$$

is the subscale (turbulent) contribution to the transport of jth component of vorticity in the ith coordinate direction. Discuss the physical interpretation of each of the terms in the above balance equation.