Linear Algebra and Differential Equations (550.291)
Homework 8 Solutions

1.

(a) With $EI = 1$ and $w(x) = w_0$ the DE becomes $y^{(4)} = w_0$. Its characteristic equation is given by $r^4 = 0$ with the root $r = 0$ of multiplicity $k = 4$. Hence we have

$$y_c(x) = C_1 + C_2 x + C_3 x^2 + C_4 x^3,$$
$$y_p(x) = Ax^4.$$

With $y_p^{(4)}(x) = 24A$ we solve for $A = w_0 / 24$. Therefore the general solution is given by

$$y(x) = y_c(x) + y_p(x) = C_1 + C_2 x + C_3 x^2 + C_4 x^3 + \frac{w_0}{24} x^4.$$

(b) For the beam of length L to be clamped at both ends we must have

$$y(0) = 0, \ y'(0) = 0, \ y(L) = 0 \text{ and } \ y'(L) = 0.$$

(c) We impose the boundary conditions of part (b) to the general solution found in part (a). First two conditions give $C_1 = C_2 = 0$, which yield the following together with the second two conditions:

$$y(L) = C_3 L^2 + C_4 L^3 + \frac{w_0}{24} L^4 = 0 \Rightarrow C_3 + C_4 L + \frac{w_0}{24} L^2 = 0, \text{ since } L > 0.$$

$$y'(L) = 2C_3 L + 3C_4 L^2 + \frac{w_0}{6} L^3 = 0 \Rightarrow 2C_3 + 3C_4 L + \frac{w_0}{6} L^2 = 0, \text{ since } L > 0.$$

Solving these two equations with two unknowns we also obtain $C_3 = w_0 L^2 / 24$ and $C_4 = -w_0 L / 12$. Thus the specific solution is given by

$$y(x) = -\frac{w_0 L}{12} x^2 + \frac{w_0 L^2}{24} x^3 + \frac{w_0}{24} x^4.$$

2.

(a) From the given DE we get the characteristic equation $0.2r^2 + 1.2r + 2 = 0$ which solves for $r_1 = -3 + i$ and $r_2 = -3 - i$. Hence we have

$$x_c(t) = e^{-3t}(C_1 \cos t + C_2 \sin t),$$
$$x_p(t) = A\cos 4t + B\sin 4t.$$

Putting $x_p(t)$ and its derivatives into the DE we solve for $A = -0.25$ and $B = 0.98$. Thus the general solution is given by

$$x(t) = x_c(t) + x_p(t) = e^{-3t}(C_1 \cos t + C_2 \sin t) - 0.25\cos 4t + 0.98\sin 4t$$

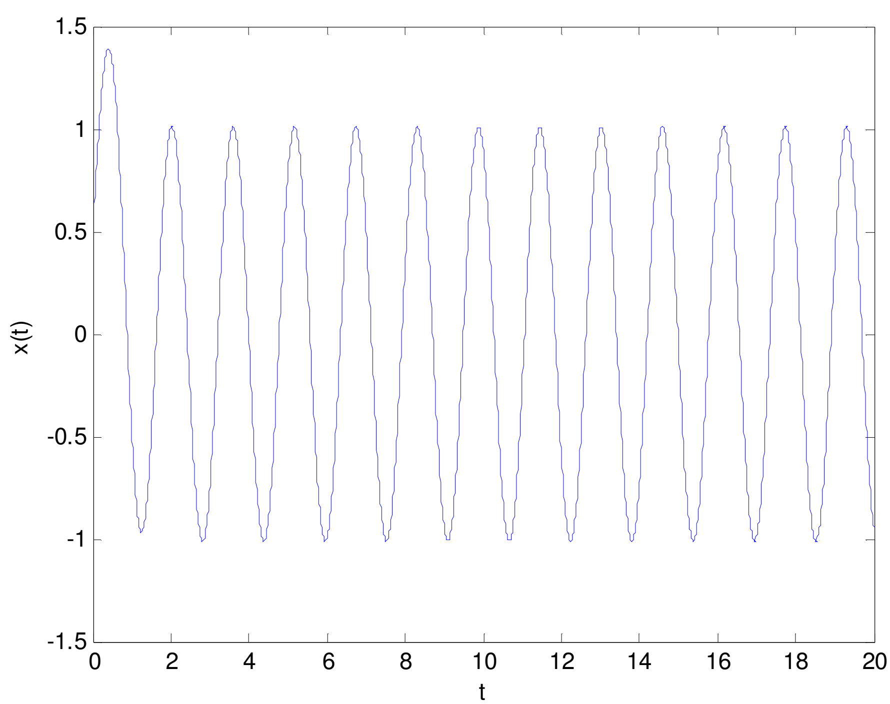
where the initial conditions yield $C_1 = 0.75$ and $C_2 = 1.69$. So the solution is given by

$$x(t) = e^{-3t}(0.75\cos t + 1.69\sin t) - 0.25\cos 4t + 0.98\sin 4t.$$

(b) As $t \to \infty$, $x_c(t)$ dies out due to the exponential and we get

$$x(t) \to x_p(t) = -0.25\cos 4t + 0.98\sin 4t.$$

(c) See figure below.



3.

(a) From the given DE we get the characteristic equation $r^2 + 9 = 0$ which solves for $r_1 = 3i$ and $r_2 = -3i$. Hence we have

$$x_c(t) = C_1 \cos 3t + C_2 \sin 3t \,,$$
$$x_p(t) = At \cos 3t + Bt \sin 3t \,.$$

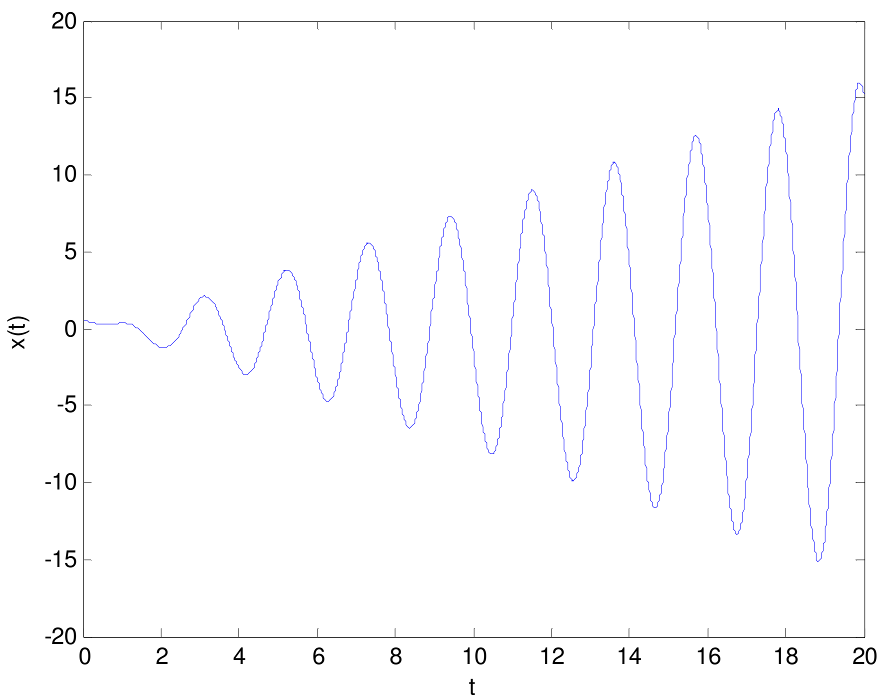
Putting $x_p(t)$ and its derivatives into the DE we solve for $A = -0.83$ and $B = 0$. Thus the general solution is given by

$$x(t) = x_c(t) + x_p(t) = C_1 \cos 3t + C_2 \sin 3t - 0.83t \cos 3t$$

where the initial conditions yield $C_1 = 0.5$ and $C_2 = 0.28$. So the solution is given by

$$x(t) = 0.5 \cos 3t + 0.28 \sin 3t - 0.83t \cos 3t$$

(b) See figure below.



(c) The ODE corresponds to a forced, undamped harmonic motion of a mass-spring system driven by an external force $5 \sin 3t$, with spring constant $k = 9$ N/m and attached mass $m = 1$ kg (such that the angular frequency of oscillations is $\sqrt{k/m} = 3\,\text{sec}^{-1}$). The initial conditions tell that the system is released from rest at $x(0) = 0.5$ m below the equilibrium point.

4.

(a) With $L = 2\,\text{ft}$ and $g = 32\,\text{ft/sec}^2$ the DE becomes $\theta'' + 16\theta = 0$. Its characteristic equation is given by $r^2 + 16 = 0$ with roots $r_1 = 4i$ and $r_2 = -4i$. Hence we have

$$\theta_c(t) = C_1 \cos 4t + C_2 \sin 4t\,,$$
$$\theta_p(t) = 0\,.$$

Thus the general solution is given by

$$\theta(t) = \theta_c(t) + \theta_p(t) = C_1 \cos 4t + C_2 \sin 4t$$

where the initial conditions yield $C_1 = 0.5$ and $C_2 = \sqrt{3}/2$. So the solution is given by

$$\theta(t) = 0.5\cos 4t + \sqrt{3}/2\sin 4t$$

(b) For the instants at which $\theta(t)$ goes to zero we solve $0.5\cos 4t + \sqrt{3}/2\sin 4t = 0$ for t. Thus we obtain

$$1 = -\sqrt{3}\tan 4t \Rightarrow t = \frac{\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)}{4} = -\frac{\pi}{24} + n\frac{\pi}{4}$$

where n is an integer.

(c) For the instants at which $\theta(t)$ attains its maximum, we solve $\theta'(t) = -2\sin 4t + 2\sqrt{3}\cos 4t = 0$ for t where $\theta''(t) = -8\cos 4t - 8\sqrt{3}\sin 4t < 0$. Thus we obtain

$$\sqrt{3} = \tan 4t\,,\ \cos 4t + \sqrt{3}\sin 4t > 0\ \Rightarrow t = \frac{\tan^{-1}\sqrt{3}}{4} = \frac{\pi}{12} + 2n\frac{\pi}{4}$$

where n is an integer.

(d) For the instants at which $\theta(t)$ attains its minimum, we solve $\theta'(t) = -2\sin 4t + 2\sqrt{3}\cos 4t = 0$ for t where $\theta''(t) = -8\cos 4t - 8\sqrt{3}\sin 4t > 0$. Thus we obtain

$$\sqrt{3} = \tan 4t\,,\ \cos 4t + \sqrt{3}\sin 4t < 0\ \Rightarrow t = \frac{\tan^{-1}\sqrt{3}}{4} = \frac{\pi}{12} + (2n-1)\frac{\pi}{4}$$

where n is an integer.

5.

(a) With $L = 5/3$ henry, $R = 10$ ohms, $C = 1/30$ farad and $E = 300$ volts the DE becomes $5/3q'' + 10q' + 30q = 300$. Its characteristic equation is $5/3r^2 + 10r + 30 = 0$ with roots $r_1 = -3 + 3i$ and $r_2 = -3 - 3i$. Hence we have

$$q_c(t) = e^{-3t}(C_1 \cos 3t + C_2 \sin 3t),$$
$$q_p(t) = A.$$

Putting $q_p(t)$ and its derivatives into the DE we solve for $A = 10$. Thus the general solution is given by

$$q(t) = q_c(t) + q_p(t) = e^{-3t}(C_1 \cos 3t + C_2 \sin 3t) + 10$$

where the initial conditions yield $C_1 = C_2 = -10$. So the solution is given by

$$q(t) = e^{-3t}(-10 \cos 3t - 10 \sin 3t) + 10.$$

(b) For the current we have

$$i(t) = q'(t) = 60e^{-3t} \sin 3t.$$

(c) For the maximum charge on the capacitor, we first solve $i(t) = q'(t) = -60e^{-3t} \sin 3t = 0$ for $t \geq 0$ where $q''(t) = 180e^{-3t} \cos 3t - 180e^{-3t} \sin 3t < 0$. Thus we obtain

$$\sin 3t = 0,\ \cos 3t < 0 \ \Rightarrow t = (2n+1)\frac{\pi}{3}$$

where n is a non-negative integer. Therefore the maximum charge for $t \geq 0$ is given by

$$\sup_{t \geq 0} q(t) = \sup_{n \geq 0} q\big((2n+1)\pi/3\big) = \sup_{n \geq 0}(10e^{-(2n+1)\pi} + 10) = 10e^{-\pi} + 10 = 10.43 \text{ coulombs}.$$

6.

(a) Since the given problem is an eigenvalue problem, the principal directions of the deformation will be the eigenvectors of the given transformation matrix $\mathbf{A}$. Solving for its spectral components we get

$$\lambda_1 = 8,\ \mathbf{v_1} = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix},$$
$$\lambda_2 = 2,\ \mathbf{v}_2 = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}.$$

Thus the principal directions are $\mathbf{v_1}$ and $\mathbf{v}_2$.

(b) The membrane is stretched by $\lambda_1 = 8$ in the first principal direction $\mathbf{v_1}$ and by $\lambda_2 = 2$ in the second principal direction $\mathbf{v}_2$. (The deformation is stretching in both principal directions since both eigenvalues are larger than 1.)

(c) The angles that principal directions make with respect to the horizontal x_1 axis are given by

$$\theta_1 = \tan^{-1}\sqrt{2} = 54.74^\circ,$$
$$\theta_2 = \tan^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -35.26^\circ$$

(d) See figure below.