

Linear Algebra and Differential Equations (550.291)
Homework 7 (Due Thursday, November 10, 2005)

General Directions: You must show all work and document any assumptions to receive full credit. All problems are to be done by hand unless otherwise stated.

1. Edwards & Penney: Problem 5.1.51

Solution:

(a)

Let $v = \ln x$ with $x > 0$. Then $\frac{dv}{dx} = \frac{1}{x}$. But, according to the chain rule,

$$\frac{dy}{dx} = \frac{dy}{dv} \frac{dv}{dx} = \frac{1}{x} \cdot \frac{dy}{dv}$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\frac{1}{x} \cdot \frac{dy}{dv} \right).$$

Using the product rule,

$$\frac{d}{dx} \left(\frac{1}{x} \cdot \frac{dy}{dv} \right) = -\frac{1}{x^2} \cdot \frac{dy}{dv} + \frac{1}{x} \cdot \frac{d}{dx} \left(\frac{dy}{dv} \right) = -\frac{1}{x^2} \cdot \frac{dy}{dv} + \frac{1}{x} \cdot \frac{d}{dv} \left(\frac{dy}{dv} \right) \cdot \frac{dv}{dx} = -\frac{1}{x^2} \cdot \frac{dy}{dv} + \frac{1}{x^2} \cdot \frac{d^2y}{dv^2}.$$

Substituting these two derivatives in the given second-order Euler equation,

$$ax^2 \left(-\frac{1}{x^2} \cdot \frac{dy}{dv} + \frac{1}{x^2} \cdot \frac{d^2y}{dv^2} \right) + bx \left(\frac{1}{x} \cdot \frac{dy}{dv} \right) + cy = 0,$$

which simplifies to

$$a \frac{d^2y}{dv^2} + (b - a) \frac{dy}{dv} + cy = 0.$$

(b)

Let r_1 and r_2 be two real and distinct roots of the modified Euler equation obtained above. Then the solution is

$$y(x) = c_1 e^{r_1 v(x)} + c_2 e^{r_2 v(x)}, \quad x > 0.$$

But $v(x) = \ln x$ implies

$$y(x) = c_1 e^{r_1 \ln(x)} + c_2 e^{r_2 \ln(x)} = c_1 x^{r_1} + c_2 x^{r_2}, \quad x > 0.$$

2. Edwards & Penney: Problem 5.1.52

Solution:

Substituting $v = \ln x$, with $x > 0$, the given differential equation can be rewritten as

$$\frac{d^2y}{dv^2} + (1 - 1)\frac{dy}{dv} - y = \frac{d^2y}{dv^2} - y = 0.$$

Then the characteristic equation $r^2 - 1 = 0$ has two distinct real roots 1 and -1 . Therefore the general solution is

$$y(x) = c_1x^1 + c_2x^{-1} = c_1x + \frac{c_2}{x}, \quad x > 0.$$

3. Edwards & Penney: Problem 5.2.21

Solution:

Since $y(x) = y_c(x) + y_p(x) = c_1 \cos x + c_2 \sin x + 3x$, $y'(x) = -c_1 \sin x + c_2 \cos x + 3$. Then $y(0) = 2$ implies $c_1 = 2$ and $y'(0) = -2$ implies that $c_2 = -5$. Therefore,

$$y(x) = 2 \cos x - 5 \sin x + 3x.$$

4. Edwards & Penney: Problem 5.2.36

Solution:

Suppose $y_1(x)$ is a solution for the homogeneous second-order linear differential equation

$$y'' + p(x)y' + q(x)y = 0.$$

By substituting $y = v(x)y_1(x)$,

$$y' = v'y_1 + vy_1',$$

and

$$y'' = v''y_1 + v'y_1' + v'y_1' + vy_1'' = v''y_1 + 2v'y_1' + vy_1''.$$

Substituting in the original second-order linear differential equation,

$$(v''y_1 + 2v'y_1' + vy_1'') + p(v'y_1 + vy_1') + q(vy_1) = 0.$$

Collecting derivatives of v with same degree,

$$v''y_1 + v'(2y_1' + py_1) + v(y_1'' + py_1' + qy_1) = 0.$$

Since y_1 is a solution to the given second-order differential equation,

$$y_1'' + py_1' + qy_1 = 0.$$

Therefore,

$$v''y_1 + v'(2y_1' + py_1) = 0.$$

Since y_1 is known, the above separable differential equation can be solved for v as follows.

Let $u = v' = \frac{dv}{dx}$, then $v'' = \frac{du}{dx}$ and

$$\begin{aligned}\frac{du}{dx}y_1 + u(2y_1' + py_1) &= 0 \\ \Rightarrow \frac{1}{u} \cdot \frac{du}{dx} &= -2\frac{y_1'}{y_1} - p \\ \Rightarrow \int \frac{du}{u} &= \int \left(-2\frac{y_1'}{y_1} - p\right) dx.\end{aligned}$$

Since $y_1'(x) = \frac{dy_1}{dx}$,

$$\begin{aligned}\ln(u) &= -2 \int \frac{1}{y_1} \frac{dy_1}{dx} dx - \int p(x) dx + A \\ &= -2 \int \frac{dy_1}{y_1} dx - \int p(x) dx + A \\ &= -2 \ln y_1 - \int p(x) dx + A,\end{aligned}$$

where A is a constant of integration.

Raising both sides as powers of e ,

$$u = C \cdot \frac{e^{-\int p(x) dx}}{y_1^2},$$

where $C = e^A$.

Since $u = \frac{dv}{dx}$,

$$\begin{aligned}\int dv &= C \int \frac{e^{-\int p(x) dx}}{y_1^2} dx \\ \Rightarrow v(x) &= C \int \frac{e^{-\int p(x) dx}}{y_1^2} dx + K,\end{aligned}$$

where K is another constant of integration.

Therefore,

$$y_2(x) = y(x) = v(x)y_1(x).$$

Since we are finding another basis solution y_2 so that $y = c_1y_1 + c_2y_2$ is a general

solution, constants of integrations C and K can be chosen at our convenience as long as $v(x)$ remains as a non-constant function. Set $C = 1$ and $K = 0$. Then

$$y_2(x) = y_1(x) \int \frac{e^{-\int p(x)dx}}{y_1^2} dx.$$

5. Edwards & Penney: Problem 5.2.37

Solution:

The homogeneous second-order linear differential equation

$$x^2 y'' - 5xy' + 9y = 0 \quad (x > 0), \tag{1}$$

and one of its basis solution $y_1(x) = x^3$ are given.

Let $y = v(x)y_1(x) = vx^3$. Then $y' = v'x^3 + 3vx^2$ and $y'' = v''x^3 + 3v'x^2 + 3v'x^2 + 6vx = v''x^3 + 6v'x^2 + 6vx$. Substituting in (1),

$$\begin{aligned} x^2(v''x^3 + 6v'x^2 + 6vx) - 5x(v'x^3 + 3vx^2) + 9(vx^3) &= 0 \\ x^5v'' + (6x^4 - 5x^4)v' + (6x^3 - 15x^3 + 9x^3)v &= 0 \\ x^5v'' + x^4v' &= 0 \\ xv'' + v' &= 0 \\ \frac{v''}{v'} &= -\frac{1}{x}. \end{aligned}$$

Let $u = v' = \frac{dv}{dx}$. Then,

$$\begin{aligned} \frac{1}{u} \cdot \frac{du}{dx} &= -\frac{1}{x} \\ \int \frac{du}{u} &= -\int \frac{dx}{x} \\ \ln u &= -\ln x + A \\ u &= \frac{C}{x}, \end{aligned}$$

where $C = e^A$ is a constant of integration.

Since $u = \frac{dv}{dx}$,

$$\frac{dv}{dx} = \frac{C}{x} \Rightarrow \int dv = C \int \frac{dx}{x} \Rightarrow v = C \ln x + K,$$

where K is another constant of integration.

With $C = 1$ and $K = 0$, $v(x) = \ln x$ is the desired non-constant function and $y_2(x) = y_1(x)v(x) = x^3 \ln x$ is the other basis solution to (1) we are pursuing.

6. Edwards & Penney: Problem 5.3.7

Solution:

The corresponding characteristic equation for the given homogeneous second-order linear differential equation is

$$4r^2 - 12r + 9 = (2r - 3)^2 = 0.$$

Therefore it has two repeated real roots $r_1 = r_2 = 3/2$ and hence the general solution is

$$y(x) = c_1 e^{3x/2} + c_2 x e^{3x/2}.$$

7. Edwards & Penney: Problem 5.3.18

Solution:

The given homogeneous second-order linear differential equation has the characteristic equation

$$r^4 = 16 \Rightarrow r^4 - 16 = 0 \Rightarrow (r^2 + 4)(r + 2)(r - 2) = 0,$$

which has two distinct real roots (2 and -2) and a pair of complex roots ($\pm 2i$). Therefore the general solution is

$$\begin{aligned} y(x) &= c_1 e^{2x} + c_2 e^{-2x} + e^{(0 \cdot x)}(c_3 \cos 2x + c_4 \sin 2x) \\ &= c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos 2x + c_4 \sin 2x. \end{aligned}$$

8. Edwards & Penney: Problem 5.3.22

Solution:

The corresponding characteristic equation of the given homogeneous second-order linear differential equation is

$$9r^2 + 6r + 4 = 0.$$

Using the quadratic formula, the roots are found to be

$$\frac{-6 \pm \sqrt{6^2 - 4(9)(4)}}{2(9)} = -\frac{1}{3} \pm \frac{i}{\sqrt{3}}.$$

Therefore the general solution is

$$y(x) = e^{-x/3} \left[c_1 \cos \left(\frac{x}{\sqrt{3}} \right) + c_2 \sin \left(\frac{x}{\sqrt{3}} \right) \right].$$