**Linear Algebra and Differential Equations (550.291)**
**Homework 4 Solutions**

1. **Edwards & Penney: Problem 3.5.11**

$$[\mathbf{A}|\mathbf{I}] = \begin{bmatrix} 1 & 5 & 1 & 1 & 0 & 0 \\ 2 & 5 & 0 & 0 & 1 & 0 \\ 2 & 7 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & -5 & -2 & -2 & 1 & 0 \\ 2 & 7 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 - 2R_1} \begin{bmatrix} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & -5 & -2 & -2 & 1 & 0 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 + R_3} \begin{bmatrix} 1 & 2 & 0 & -1 & 0 & 1 \\ 0 & -5 & -2 & -2 & 1 & 0 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 - 2R_3} \begin{bmatrix} 1 & 2 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 + 3R_2} \begin{bmatrix} 1 & 2 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & -1 & 4 & 3 & -5 \end{bmatrix}$$

$$\xrightarrow{(-1)R_3} \begin{bmatrix} 1 & 2 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & -4 & -3 & 5 \end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 0 & -5 & -2 & 5 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & -4 & -3 & 5 \end{bmatrix}$$

thus $\mathbf{A}^{-1} = \begin{bmatrix} -5 & -2 & 5 \\ 2 & 1 & -2 \\ -4 & -3 & 5 \end{bmatrix}$

2. $\det \mathbf{A} = (1)\begin{vmatrix} 2 & 5 \\ 2 & 7 \end{vmatrix} + (1)\begin{vmatrix} 1 & 5 \\ 2 & 5 \end{vmatrix} = (14 - 10) + (5 - 10) = -1$

3. **Edwards & Penney: Problem 3.6.3**

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 5 & 0 \\ 3 & 6 & 9 & 8 \\ 4 & 0 & 10 & 7 \end{vmatrix} = (1)\begin{vmatrix} 1 & 5 & 0 \\ 6 & 9 & 8 \\ 0 & 10 & 7 \end{vmatrix} = (-5)\begin{vmatrix} 6 & 8 \\ 0 & 7 \end{vmatrix} = (-5)(42 - 0) = -210$$

4. **Edwards & Penney: Problem 3.6.27**

$$\triangle = \begin{vmatrix} 5 & 2 & 2 \\ 1 & 5 & -3 \\ 5 & -3 & 5 \end{vmatrix} = 96; \quad x_1 = \frac{1}{\triangle}\begin{vmatrix} 1 & 2 & 2 \\ -2 & 5 & -3 \\ 2 & -3 & 5 \end{vmatrix} = \frac{1}{3},$$

$$x_2 = \frac{1}{\triangle}\begin{vmatrix} 5 & 1 & 2 \\ 1 & -2 & -3 \\ 5 & 2 & 5 \end{vmatrix} = -\frac{2}{3}, \quad x_3 = \frac{1}{\triangle}\begin{vmatrix} 5 & 2 & 1 \\ 1 & 5 & -2 \\ 5 & -3 & 2 \end{vmatrix} = -\frac{1}{3}$$

5. **Edwards & Penney: Problem 3.6.33**

$$\det \mathbf{A} = \begin{vmatrix} -5 & -2 & 2 \\ 1 & 5 & -3 \\ 5 & -3 & 1 \end{vmatrix} = (-5)\begin{vmatrix} 5 & -3 \\ -3 & 1 \end{vmatrix} - (-2)\begin{vmatrix} 1 & -3 \\ 5 & 1 \end{vmatrix} + (2)\begin{vmatrix} 1 & 5 \\ 5 & -3 \end{vmatrix} = -4;$$

$$A_{11} = \begin{vmatrix} 5 & -3 \\ -3 & 1 \end{vmatrix} = -4, \quad A_{12} = -\begin{vmatrix} 1 & -3 \\ 5 & 1 \end{vmatrix} = -16, \quad A_{13} = \begin{vmatrix} 1 & 5 \\ 5 & -3 \end{vmatrix} = -28,$$

$$A_{21} = -\begin{vmatrix} -2 & 2 \\ -3 & 1 \end{vmatrix} = -4, \quad A_{22} = \begin{vmatrix} -5 & 2 \\ 5 & 1 \end{vmatrix} = -15, \quad A_{23} = -\begin{vmatrix} -5 & -2 \\ 5 & -3 \end{vmatrix} = -25,$$

$$A_{31} = \begin{vmatrix} -2 & 2 \\ 5 & -3 \end{vmatrix} = -4, \quad A_{32} = -\begin{vmatrix} -5 & 2 \\ 1 & -3 \end{vmatrix} = -13, \quad A_{33} = \begin{vmatrix} -5 & -2 \\ 1 & 5 \end{vmatrix} = -23;$$

$$\mathbf{A}^{-1} = \frac{[A_{ij}]^T}{\det \mathbf{A}} = \frac{1}{-4}\begin{bmatrix} -4 & -4 & -4 \\ -16 & -15 & -13 \\ -28 & -25 & -23 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 15/4 & 13/4 \\ 7 & 25/4 & 23/4 \end{bmatrix}$$

6. **Edwards & Penney: Problem 3.6.50**

   If $\mathbf{A}^2 = \mathbf{A}$ then $|\mathbf{A}|^2 = |\mathbf{A}|$, so $|\mathbf{A}|^2 - |\mathbf{A}| = |\mathbf{A}|(|\mathbf{A}| - 1) = 0$, and hence it follows immediately that either $|\mathbf{A}| = 0$ or $|\mathbf{A}| = 1$.

7. **Edwards & Penney: Problem 4.2.5**

   Suppose $\mathbf{x} = (x_1, x_2, x_3, x_4)$ and $\mathbf{y} = (y_1, y_2, y_3, y_4)$ are vectors in $W$, so

$$x_1 + 2x_2 + 3x_3 + 4x_4 = 0 \quad \text{and} \quad \mathrm{y}_1 + 2\mathrm{y}_2 + 3\mathrm{y}_3 + 4\mathrm{y}_4 = 0.$$

   Then their sum $\mathbf{s} = \mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4) = (s_1, s_2, s_3, s_4)$ satisfies the same condition

$$\begin{aligned} s_1 + 2s_2 + 3s_3 + 4s_4 &= (x_1 + y_1) + 2(x_2 + y_2) + 3(x_3 + y_3) + 4(x_4 + y_4) \\ &= (x_1 + 2x_2 + 3x_3 + 4x_4) + (y_1 + 2y_2 + 3y_3 + 4y_4) = 0 + 0 = 0 \end{aligned}$$

   and thus is an element of $W$. Similarly, the scalar multiple $\mathbf{m} = c\mathbf{x} = (cx_1, cx_2, cx_3, cx_4) = (m_1, m_2, m_3, m_4)$ satisfies the condition

$$m_1 + 3m_2 + 3m_3 + 4m_4 = cx_1 + 2cx_2 + 3cx_3 + 4cx_4 = c(x_1 + 2x_2 + 3x_3 + 4x_4) = 0$$

   and hence is an element of $W$. Therefore $W$ is a subspace of $\mathbf{R}^4$.

8. **Edwards & Penney: Problem 4.2.15**

$$\mathbf{A} = \begin{bmatrix} 1 & -4 & 1 & -4 \\ 1 & 2 & 1 & 8 \\ 1 & 1 & 1 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$$

   Thus $x_3 = s$ and $x_4 = t$ are free variables. We solve for $x_1 = -s - 4t$ and $x_2 = -2t$, so

$$\begin{aligned} \mathbf{x} = (x_1, x_2, x_3, x_4) &= (-s - 4t, -2t, s, t) \\ &= (-s, 0, s, 0) + (-4t, -2t, 0, t) = s\mathbf{u} + t\mathbf{v} \end{aligned}$$

   where $\mathbf{u} = (-1, 0, 1, 0)$ and $\mathbf{v} = (-4, -2, 0, 1)$.

9. **Edwards & Penney: Problem 4.3.19**

$$\mathbf{A} = \begin{bmatrix} 2 & 5 & 2 \\ 0 & 4 & -1 \\ 3 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$$

We see that the system of 4 equations in 3 unknowns has the unique solution $c_1 = c_2 = c_3 = 0$, so the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly independent.

10. **Edwards & Penney: Problem 4.3.21**

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{E}$$

We see that the system of 4 equations in 3 unknowns has a 1-dimensional solution space. If we choose $c_3 = -1$ then $c_1 = 1$ and $c_2 = -2$. Therefore $\mathbf{v}_1 - 2\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$.