Solution HW10 (550.291)

1) $\underline{\underline{8.1.7}}$

$\lambda_1 = 0, \quad v_1 = [6 \ 2 \ 5]^T$

$\lambda_2 = 1, \quad v_2 = [3 \ 1 \ 2]^T$

$\lambda_3 = -1, \quad v_3 = [2 \ 1 \ 2]^T$

$$\Phi(t) = [e^{\lambda_1 t} v_1 \quad e^{\lambda_2 t} v_2 \quad e^{\lambda_3 t} v_3]$$

$$= \begin{bmatrix} 6 & 3e^t & 2e^{-t} \\ 2 & e^t & e^{-t} \\ 5 & 2e^t & 2e^{-t} \end{bmatrix}$$

$$x(t) = \begin{bmatrix} 6 & 3e^t & 2e^{-t} \\ 2 & e^t & e^{-t} \\ 5 & 2e^t & 2e^{-t} \end{bmatrix} \begin{bmatrix} 0 & -2 & 1 \\ 1 & 2 & -2 \\ -1 & 3 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -12 + 12e^t + 2e^{-t} \\ -4 + 4e^t + e^{-t} \\ -10 + 8e^t + 2e^{-t} \end{bmatrix}$$

2) 8.1.18

Eigensystem: $\lambda_1 = 2$, $v_1 = [1\ \ -1]^T$

$\lambda_2 = 6$, $v_2 = [1\ \ 1]^T$

$$\Phi(t) = [e^{\lambda_1 t} v_1 \quad e^{\lambda_2 t} v_2] = \begin{bmatrix} e^{2t} & e^{6t} \\ -e^{2t} & e^{6t} \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} e^{2t} & e^{6t} \\ -e^{2t} & e^{6t} \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{2t}+e^{6t} & -e^{2t}+e^{6t} \\ -e^{2t}+e^{6t} & e^{2t}+e^{6t} \end{bmatrix}$$

3) 8.2.13

The coefficient matrix of the associated homogeneous system has eigen values $\lambda_1 = 1$ and $\lambda_2 = 3$, so there is duplication of e^t terms. We therefore substitute the trial solution

$$x_p(t) = (a_1 + b_1 t)e^t, \quad y_p(t) = (a_2 + b_2 t)e^t$$

This leads to the particular solution

$$x(t) = \frac{1}{2}(1+5t)e^t, \quad y(t) = -\frac{5}{2} t e^t$$

4) 8.2.17

$$x_1(t) = 102 - 95e^{-t} - 7e^{5t}$$

$$x_2(t) = 96 - 95e^{-t} - e^{5t}$$

5) 10.2.14

The transformed equations are:

$$s^2 X(s) + 1 + 2X(s) + 4Y(s) = 0$$

$$s^2 Y(s) + 1 + X(s) + 2Y(s) = 0$$

which we solve for,

$$X(s) = \frac{-s^2+2}{s^2(s^2+4)} = \frac{1}{4}\left[2 \cdot \frac{1}{s^2} - 3 \cdot \frac{2}{s^2+4}\right]$$

$$Y(s) = \frac{-s^2-1}{s^2(s^2+4)} = -\frac{1}{8}\left[2 \cdot \frac{1}{s^2} + 3 \cdot \frac{2}{s^2+4}\right]$$

Hence the solution is:

$$x(t) = \frac{1}{4}(2t - 3\sin 2t)$$

$$y(t) = \left(-\frac{1}{8}\right)(2t + 3\sin 2t)$$

6) 10.3.37

$$[s^2 X(s) - 2] + 4s\,X(s) + 13X(s) = \frac{1}{(s+1)^2}$$

$$X(s) = \frac{2 + 1/(s+1)^2}{s^2+4s+13}$$

$$= \frac{2s^2+4s+13}{(s+1)^2(s^2+4s+13)}$$

$$= \frac{1}{50}\left[-\frac{1}{s+1} + \frac{5}{(s+1)^2} + \frac{s+98}{(s+2)^2+9}\right]$$

$$= \frac{1}{50}\left[-\frac{1}{s+1} + \frac{5}{(s+1)^2} + \frac{s+2}{(s+2)^2+9} + 32\cdot\frac{3}{(s+2)^2+9}\right]$$

7) (a) $F(s) = \int_{-\infty}^{0} e^{-st}\,0\,dt + \int_0^1 e^{-st}t\,dt + \int_1^2 e^{-st}(2-t)\,dt$

$$+\int_2^\infty e^{-st}\,0\,dt$$

$$= \int_0^1 e^{-st}t\,dt + \int_1^2 e^{-st}(2-t)\,dt$$

$$= \frac{-\cancel{se^{-s}} - e^{-s} + 1 - 2\cancel{s}e^{-2s} + \cancel{2se^{-s}} + \cancel{2se^{-2s}} + e^{-2s} - \cancel{se^{-s}} - e^{-s}}{s^2}$$

$$= \frac{e^{-2s} - 2e^{-s} + 1}{s^2}$$

(e)

$$F(s) = \int_1^3 e^{-st}(t-1)\,dt + \int_3^\infty e^{-st}(2)\,dt$$

$$= \int_1^3 te^{-st}\,dt - \int_1^3 e^{-st}\,dt + \int_3^\infty 2e^{-st}\,dt$$

$$= \left[\frac{e^{-st}(-st-1)}{s^2}\right]_1^3 - \left[\frac{e^{-st}}{-s}\right]_1^3 + \left[\frac{2e^{-st}}{-s}\right]_3^\infty$$

$$= \frac{e^{-3s}(-3s-1) - e^{-s}(-s-1) + se^{-3s} - se^{-s} + s2e^{-3s}}{s^2}$$

$$= \frac{-e^{-3s} + e^{-s}}{s^2}$$